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4.4 Change of Basis

Standard basis for $\mathbb{R}^n$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, ..., $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Fact If $A \in \mathbb{R}^{n \times n}$ is such that A^{-1} exists, then the columns of A necessarily form a basis for $\mathbb{R}^n$

Pf: A has n-columns that are linearly independent. Since $\dim(\mathbb{R}^n) = n$, the columns of A must form a basis.

$\mathcal{S}_n = \{e_1, \ldots, e_n\}$ the standard basis for $\mathbb{R}^n$

(104) Does $\mathcal{B} = \left\{ \overset{u^1}{\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}}, \overset{u^2}{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}, \overset{u^3}{\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}} \right\}$ form a basis for $\mathbb{R}^3$?

$\bar{U}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 3 & 6 & 5 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -6 & -7 & -3 & 1 & 0 \\ 0 & 5 & 7 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 7 & -3 & 5 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -1 & 2 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -3/7 & \frac{5}{7} & \frac{6}{7} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/7 & \frac{-6}{7} & \frac{-10}{7} \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -3/7 & 5/7 & 6/7 \end{array}\right]$$

$$\bar{U}^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -6 & -10 \\ 7 & -7 & -7 \\ -3 & 5 & 6 \end{bmatrix}$$

Given $y \in \mathbb{R}^3$ find $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$y = x_1 u^1 + x_2 u^2 + x_3 u^3$$

$$= [u^1 \ u^2 \ u^3]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \bar{U} x$$

$$\Rightarrow \quad x = \bar{U}^{-1} y = [y]_{\mathcal{B}}$$

We call the components of x the coordinates of y in the basis $\mathcal{B}$

105 **ex**: what are the coordinates of

$[Y]_{\mathcal{E}} = Y$ $\qquad Y = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ in the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right\}$?

$$X = U^{-1} Y = \frac{1}{7} \begin{bmatrix} 5 & -6 & -10 \\ 7 & -7 & -7 \\ -3 & 5 & 6 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{7} \begin{bmatrix} 10 + 6 + 10 \\ 14 + 7 + 7 \\ -6 - 5 - 6 \end{bmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 26 \\ 28 \\ -17 \end{pmatrix} = \begin{pmatrix} 26/7 \\ 4 \\ -17/7 \end{pmatrix}$$

Notation: $[Y]_{\mathcal{B}} = \begin{bmatrix} 26/7 \\ 4 \\ -17/7 \end{bmatrix} = U^{-1} Y$, $U = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 5 \\ -2 & 1 & -1 \end{bmatrix}$

$\Downarrow$

The vector that gives Y in the $\mathcal{B}$ coordinates.

<u>Conversion</u>: $[Y]_{\mathcal{E}_n} = Y$

$$\left[\begin{array}{ccc|c|ccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|ccc} 1 & 0 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -3 & 1 & 4 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|ccc} 1 & 0 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|ccc} 1 & 0 & 0 & 2 & -6 & -2 & 7 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 & 1 & -3 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix}$$

$$= \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix} \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = \begin{bmatrix} -18 - 8 + 28 \\ 3 - 4 \\ 9 + 4 - 12 \end{bmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

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Suppose $\mathcal{B} = \{u^1, u^2, \dots, u^n\}$ is a basis for $\mathbb{R}^n$.

Then $\bar{U} = [u^1\, u^2 \dots u^n] \in \mathbb{R}^{n\times n}$ is invertible.

The coordinates of y in this basis are

$$x = \bar{U}^{-1} y = [y]_{\mathcal{B}}$$

This gives $y = \bar{U} x$

ex: Let $\mathcal{B} = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-3\\0\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix} \right\}$ what are the coordinates of $y = \begin{pmatrix}3\\4\\4\end{pmatrix}$ in this basis

$\bar{U}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & -2 & 7 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 & 1 & -3 \end{array}\right]$$

$\bar{U}^{-1}$

check

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$[y]_{\mathcal{B}} = U^{-1} \begin{pmatrix}3\\4\\4\end{pmatrix} = \begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix} \begin{pmatrix}3\\4\\4\end{pmatrix} = \begin{bmatrix} -18 - 8 + 28 \\ 3 - 4 \\ 9 + 4 - 12 \end{bmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix}$$

check

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}3\\4\\4\end{pmatrix} \checkmark$$

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Two nonstandard Bases

$$\mathcal{B}_1 = \{u^1, u^2, \dots, u^n\}, \quad \mathcal{B}_2 = \{v^1, v^2, \dots, v^n\}$$

$$\bar{U} = [u^1\ u^2 \dots u^n] \qquad \bar{V} = [v^1\ v^2 \dots v^n]$$

Given $y \in \mathbb{R}^n$, how do we get from $[y]_{\mathcal{B}_1}$ to $[y]_{\mathcal{B}_2}$?
That is how do we change the coordinate representation of y in $\mathcal{B}_1$ to the coordinates of y in $\mathcal{B}_2$?

Recall $[y]_{\mathcal{B}_1} = \bar{U}^{-1} y$ and $[y]_{\mathcal{B}_2} = \bar{V}^{-1} y$

so $\bar{V}^{-1} \bar{U} [y]_{\mathcal{B}_1} = \bar{V}^{-1} \bar{U} \bar{U}^{-1} y = \bar{V}^{-1} y = [y]_{\mathcal{B}_2}$

$\mathcal{S}$ standard basis

$$\begin{tikzcd}
\mathcal{B}_1 \arrow[r, bend left, "\bar{U}"] \arrow[rr, bend left=60, "\bar{V}^{-1}\bar{U}"] & \mathcal{S} \arrow[l, bend left, "\bar{U}^{-1}"] \arrow[r, bend right, "\bar{V}^{-1}"'] & \mathcal{B}_2 \arrow[l, bend right, "\bar{V}"'] \arrow[ll, bend left=60, "\bar{U}^{-1}\bar{V}"]
\end{tikzcd}$$

Change of basis matrix matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ is $\bar{V}^{-1}\bar{U}$

(108)

ex: $\mathcal{B}_1 = \left\{ \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$ $\quad \mathcal{B}_2 = \left[\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 0 \end{pmatrix} \right]$

What is the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$?

$$U = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & -1 \\ 4 & 3 & 2 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 7 \\ -1 & -1 & 0 \end{bmatrix} \Rightarrow \underbrace{V^{-1}U}_{\text{change of basis matrix}} [Y]_{\mathcal{B}_1} = [Y]_{\mathcal{B}_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 3 & 1 & 7 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -7 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

check

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 7 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -7 & 3 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow V^{-1}U = \begin{bmatrix} 7 & -3 & -3 \\ -7 & 3 & 2 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & -1 \\ 4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -19 & -4 & 4 \\ 15 & 1 & -6 \\ 6 & 2 & -1 \end{bmatrix}$$

109 Change of basis between subspaces.

Thm Let $S \subset \mathbb{R}^n$ be a subspace and let

$\mathcal{B}_1 = \{u^1, \dots, u^k\}$, $\mathcal{B}_2 = \{v^1, \dots, v^k\}$ be two bases for S.

Suppose $[u^i]_{\mathcal{B}_2}$ are the coordinates of u^i in the basis $\mathcal{B}_2$, $i = 1 \dots k$

and set $C = \big[[u^1]_{\mathcal{B}_2} \; [u^2]_{\mathcal{B}_2} \cdots [u^k]_{\mathcal{B}_2} \big] \in \mathbb{R}^{k \times k}$

Then for any $x \in S$, $[x]_{\mathcal{B}_2} = C[x]_{\mathcal{B}_1}$

Proof: Let $x \in S$ with $x = x_1 u^1 + x_2 u^2 + \dots + x_k u^k$ so that $[x]_{\mathcal{B}_1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$. Then

$$C[x]_{\mathcal{B}_1} = x_1 [u^1]_{\mathcal{B}_2} + x_2 [u^2]_{\mathcal{B}_2} + \dots + x_k [u^k]_{\mathcal{B}_2} = [x_1 u^1 + x_2 u^2 + \dots + x_k u^k]_{\mathcal{B}_2}$$
$$= [x]_{\mathcal{B}_2}$$

(11.D) ex: $\mathcal{B}_1 = \left\{ \underset{u^1}{\begin{pmatrix}1\\-5\\8\end{pmatrix}}, \underset{u^2}{\begin{pmatrix}3\\-8\\3\end{pmatrix}} \right\}$, $\mathcal{B}_2 = \left\{ \underset{v^1}{\begin{pmatrix}1\\-3\\2\end{pmatrix}}, \underset{v^2}{\begin{pmatrix}-1\\2\\1\end{pmatrix}} \right\}$

Does span $\mathcal{B}_1$ = span $\mathcal{B}_2$?

$$\begin{array}{c|ccc} v^{1T} & 1 & -3 & 2 \\ v^{2T} & -1 & 2 & 1 \\ u^{1T} & 1 & -5 & 8 \\ u^{2T} & 3 & -8 & 3 \\ \hline & 1 & -3 & 2 \\ & 0 & -1 & 3 \\ & 0 & -3 & 9 \\ & 0 & -2 & 6 \\ \hline \end{array}$$

Yes

What are $[u^1]_{\mathcal{B}_2}$ and $[u^2]_{\mathcal{B}_2}$?

$$\begin{array}{cc|cc} v^1 & v^2 & u^1 & u^2 \\ 1 & -1 & 1 & 3 \\ -3 & 2 & -5 & -8 \\ 2 & 1 & 8 & 3 \\ \hline 1 & -1 & 1 & 3 \\ 0 & -1 & -2 & 1 \\ 0 & 3 & 6 & -3 \\ \hline 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}$$

$\Rightarrow [u^1]_{\mathcal{B}_2} = \begin{pmatrix}3\\2\end{pmatrix}$, $[u^2]_{\mathcal{B}_2} = \begin{pmatrix}2\\-1\end{pmatrix}$

<u>Check</u>

$$\begin{bmatrix}1 & -1\\-3 & 2\\2 & 1\end{bmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\-5\\8\end{pmatrix} = u^1$$

$$\begin{bmatrix}1 & -1\\-3 & 2\\2 & 1\end{bmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}3\\-8\\3\end{pmatrix} = u^1$$

If $[x]_{\mathcal{B}_1} = \begin{pmatrix}3\\-1\end{pmatrix}$, what is x in $\mathcal{S}_3$?

$$\begin{bmatrix}1 & 3\\-5 & -8\\8 & 3\end{bmatrix}\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}0\\-7\\21\end{pmatrix}$$

what is $[x]_{\mathcal{B}_2}$?

$$C = \left[[u^1]_{\mathcal{B}_2}\ [u^2]_{\mathcal{B}_2} \right] = \begin{bmatrix}3 & 2\\2 & -1\end{bmatrix} \Rightarrow [x]_{\mathcal{B}_2} = C[x]_{\mathcal{B}_1} = \begin{bmatrix}3 & 2\\2 & -1\end{bmatrix}\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}7\\7\end{pmatrix}$$

<u>Check</u>: $\begin{bmatrix}1 & -1\\-3 & 2\\2 & 1\end{bmatrix}\begin{pmatrix}7\\7\end{pmatrix} = \begin{pmatrix}0\\-7\\21\end{pmatrix}$ ✓

(111) Find the change of Basis matrix for

$$\mathcal{B}_1 = \left\{ \underset{u^1}{\begin{pmatrix} -10 \\ 4 \\ 2 \\ -5 \end{pmatrix}}, \underset{u^2}{\begin{pmatrix} -5 \\ 5 \\ 0 \\ -4 \end{pmatrix}}, \underset{u^3}{\begin{pmatrix} -8 \\ 5 \\ 2 \\ -6 \end{pmatrix}} \right\} \qquad \mathcal{B}_2 = \left\{ \underset{v^1}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}}, \underset{v^2}{\begin{pmatrix} -3 \\ 0 \\ 2 \\ -2 \end{pmatrix}}, \underset{v^3}{\begin{pmatrix} 6 \\ -3 \\ 2 \\ 0 \end{pmatrix}} \right\}$$

$$\begin{array}{ccc|ccc}
v^1 & v^2 & v^3 & u^1 & u^2 & u^3 \\
2 & -3 & 6 & -10 & -5 & -8 \\
1 & 0 & -3 & 4 & 5 & 5 \\
0 & 2 & 2 & 2 & 0 & 2 \\
-1 & -2 & 0 & -5 & -4 & -6 \\
\hline
1 & 0 & -3 & 4 & 5 & 5 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & -2 & -3 & -1 & 1 & -1 \\
0 & -3 & 12 & -18 & -15 & -18 \\
\hline
1 & 0 & -3 & 4 & 5 & 5 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & -1 & 1 & 1 & 1 \\
0 & 0 & 15 & -15 & -15 & -15 \\
\hline
1 & 0 & 0 & 1 & 2 & 2 \\
0 & 1 & 0 & 2 & 1 & 2 \\
0 & 0 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

$$\Rightarrow [u^1]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad [u^2]_{\mathcal{B}_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad [u^3]_{\mathcal{B}_2} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

If $[x]_{\mathcal{B}_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

(a) what is $[x]_{\mathcal{B}_3}$? $\begin{array}{c} \begin{array}{ccc} u^1 & u^2 & u^3 \end{array} \\ \begin{bmatrix} -10 & -5 & -8 \\ 4 & 5 & 5 \\ 2 & 0 & 2 \\ -5 & -4 & -6 \end{bmatrix} \end{array} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -31 \\ 19 \\ -21 \end{pmatrix}$

(b) what is $[x]_{\mathcal{B}_2}$? $C[x]_{\mathcal{B}_1} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix}$

Check: $\begin{bmatrix} 2 & -3 & 6 \\ 1 & 0 & -3 \\ 0 & 2 & 2 \\ -1 & -2 & 0 \end{bmatrix} \begin{pmatrix} 7 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 14 - 21 - 24 \\ 7 + 12 \\ 14 - 8 \\ -7 - 14 \end{pmatrix} = \begin{pmatrix} -31 \\ 19 \\ -21 \end{pmatrix}$