(A) 11-08-22 Review

Find a linear transformation T that has

$$\ker(T) = \text{span}\left[\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\\2\end{pmatrix}\right].$$

$\ker(T_A) =$

We find a matrix A such that $\ker(T_A) = \text{Span}\left[\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\\2\end{pmatrix}\right]$

We can also choose A to be in echelon form.

$$N = \begin{bmatrix}1 & 0\\0 & 1\\-2 & 3\\1 & 2\end{bmatrix} = \begin{bmatrix}I_2\\M\end{bmatrix}, \quad M = \begin{bmatrix}-2 & 3\\1 & 2\end{bmatrix}$$

Choose $A = [-M \;\; I_2] \Rightarrow AN = [-M \;\; I]\begin{bmatrix}I\\M\end{bmatrix} = -M + M = 0$

and $AN = \left[A\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, A\begin{pmatrix}0\\1\\3\\2\end{pmatrix}\right] = [0, 0]$

Finally, nullity $A = 4 - \text{rank}(A) = 4 - 2 = 2$. $\quad A \in \mathbb{R}^{2\times 4}$

so $\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\\2\end{pmatrix}$ form a basis for $\text{Nul}(A)$.

(B) Find a linear transformation T that has

$$\ker(T) = \text{span}\left[\begin{pmatrix}-1\\3\\1\\2\end{pmatrix}, \begin{pmatrix}-3\\10\\2\\1\end{pmatrix}\right]$$

<u>Method</u>: Put $\begin{bmatrix}-1 & 3 & 1 & 2\\ -3 & 10 & 2 & 1\end{bmatrix}$ into reduced echelon form

$$\begin{matrix}1 & -3 & -1 & -2\\ 0 & 1 & -1 & -5\end{matrix}$$

$$\begin{matrix}1 & 0 & -4 & -17\\ 0 & 1 & -1 & -5\end{matrix} = [I \ M^T]$$

so $\text{Span}\left[\begin{pmatrix}-1\\3\\1\\2\end{pmatrix}, \begin{pmatrix}-3\\10\\2\\1\end{pmatrix}\right] = \text{Span}\left[\begin{pmatrix}1\\0\\-4\\-17\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-5\end{pmatrix}\right] \rightarrow \begin{bmatrix}I\\M\end{bmatrix}$

so $A = [-M \ I] = \begin{bmatrix}4 & 1 & 1 & 0\\ 17 & 5 & 0 & 1\end{bmatrix}$

check $\begin{bmatrix}4 & 1 & 1 & 0\\ 17 & 5 & 0 & 1\end{bmatrix}\begin{pmatrix}-1\\3\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, $\begin{bmatrix}4 & 1 & 1 & 0\\ 17 & 5 & 0 & 1\end{bmatrix}\begin{pmatrix}-3\\10\\2\\1\end{pmatrix} = \begin{Bmatrix}0\\-51+50+1\end{Bmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$T = T_A$

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Orthogonality

Recall from 3^{rd} term calculus that for $u, v \in \mathbb{R}^n$

$$u \cdot v = \|u\| \, \|v\| \cos\theta$$

where $\|u\| = (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}$, $\|v\| = (v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$

(Euclidean norm or magnitude of a vector)

θ = the smallest angle between u and v.

So $u \cdot v = 0 \iff u$ and v are perpendicular to each other

When $u \cdot v = 0$ we say that u and v are orthogonal to each other and write $u \perp v$.

also $u \cdot v = u^T v = (u^T v)^T = v^T u$

(99) Given $u^1, u^2, \dots, u^k \in \mathbb{R}^n$ define

$$\{u^1, u^2, \dots, u^k\}^\perp := \{v \in \mathbb{R}^n : u^i \cdot v = 0,\ i = 1, 2, \dots k\}$$

$$= \mathrm{Nul}(\bar{U}^T)$$

where $\bar{U} = [u^1\ u^2 \dots u^k]$

since $\bar{U}^T v = \begin{bmatrix} (u^1)^T \\ (u^2)^T \\ \vdots \\ (u^k)^T \end{bmatrix} v = \begin{bmatrix} (u^1)^T v \\ (u^2)^T v \\ \vdots \\ (u^k)^T v \end{bmatrix} = \begin{bmatrix} u^1 \cdot v \\ u^2 \cdot v \\ \vdots \\ u^k \cdot v \end{bmatrix}$

Therefore, $\{u^1, u^2, \dots, u^k\}^\perp$ is a subspace.

We call this subspace the subspace perpendicular or orthogonal to $\{u^1, u^2, \dots, u^k\}$.

(100) Theorem: $u^1, u^2, \ldots, u^k \in \mathbb{R}^n$, Then

$$\{u^1, u^2, \ldots, u^k\}^\perp = \left[\text{span}[u^1, u^2, \ldots, u^k]\right]^\perp$$

That is, if $u^i \cdot v = 0$ $i = 1, 2, \ldots, k$, Then $u \cdot v = 0$ for all $u \in \text{span}[u^1, u^2, \ldots, u^k]$

Proof: $u \in \text{Span}[u^1, u^2, \ldots, u^k] \Rightarrow u = \alpha_1 u^1 + \alpha_2 u^2 + \cdots + \alpha_k u^k$

so $u \cdot v = u^T v = (u^T v)^T = v^T u$

$$= v^T [\alpha_1 u^1 + \alpha_2 u^2 + \cdots + \alpha_k u^k]$$
$$= \alpha_1 v^T u^1 + \alpha_2 v^T u^2 + \cdots + \alpha_k v^T u^k$$
$$= 0$$

$\text{span}[u^1 \ldots u^k] = \text{Ran}(\bar{u})$ $\quad \bar{u} = [u^1\ u^2 \cdots u^k]$

$[\text{Ran}(\bar{u})]^\perp = [\text{span}\{u^1, \ldots, u^k\}]^\perp = \text{Nul}(\bar{u}^T)$

(101) <u>Theorem</u>: Let $S \subset \mathbb{R}^n$ be a subspace. Then $[S^\perp]^\perp = S$.

<u>Proof</u>: $[S \subset [S^\perp]^\perp]$ Let $u \in S$. Then $u \cdot v = 0$ $\forall v \in S^\perp$ so $u \in [S^\perp]^\perp$.

$[[S^\perp]^\perp \subset S]$ Let $u \in [S^\perp]^\perp$ so $u \cdot v = 0$ $\forall v \in S^\perp$.

Let $\{u^1, u^2, \dots, u^k\}$ be a basis for S ($\dim(S) = k$). $U = [u^1\ u^2 \dots u^k]$

Then $\mathrm{Nul}(U^T) = \{v : U^T v = 0\} = \{v : \begin{bmatrix} (u^1)^T \\ \vdots \\ (u^k)^T \end{bmatrix} v = 0\} = \{v : (u^i)^T v = 0\}$

$= S^\perp$.

If $u \notin S$, $u \neq 0$, then $\{u, u^1, \dots, u^k\}$ are linearly independent.

Set $\hat{U} = [u\ u^1 \dots u^k]$ so $S^\perp \subset \mathrm{Nul}(\hat{U}^T) \subset \mathrm{Nul}(U^T) = S^\perp$

$\dim(S^\perp) = \dim(\mathrm{Nul}(U^T)) = n - \dim(\mathrm{Ran}(U^T)) = n - \dim(\mathrm{Ran}(U)) = n-k$

$\dim(S^\perp) = \dim(\mathrm{Nul}(\hat{U}^T)) = n - \dim(\mathrm{Ran}(\hat{U}^T)) = n - \dim(\mathrm{Ran}(\hat{U})) = n-(k+1)$

This contradiction implies that $u \in S$ so $[S^\perp]^\perp \subset S$.

Consequently $S = [S^\perp]^\perp$.

(102) **Theorem:** [Four Fundamental Subspaces]

Let $A \in \mathbb{R}^{n \times m}$. Then

$$\mathrm{Nul}(A) = \mathrm{Ran}(A^T)^{\perp}$$

and

$$\mathrm{Ran}(A) = \mathrm{Nul}(A^T)^{\perp}$$

Proof: $A = \begin{bmatrix} (u^1)^T \\ \vdots \\ (u^n)^T \end{bmatrix} \Rightarrow \mathrm{Nul}(A) = \{u^1, \ldots, u^n\}^{\perp}$

$$= \mathrm{span}\{u^1, u^2, \ldots, u^n\}^{\perp}$$

$$= \mathrm{Ran}[u^1\ u^2, \ldots u^n]^{\perp}$$

$$= \mathrm{Ran}(A^T)^{\perp}$$

Replace A by A^T to get

$$\mathrm{Nul}(A^T) = \mathrm{Ran}((A^T)^T)^{\perp} = \mathrm{Ran}(A)^{\perp}$$

so $\mathrm{Nul}(A^T)^{\perp} = [\mathrm{Ran}(A)^{\perp}]^{\perp} = \mathrm{Ran}(A)$.

(103) 4.4 Change of Basis

Standard basis for $\mathbb{R}^n$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Fact If $A \in \mathbb{R}^n$ is such that A^{-1} exists, then the columns of A necessarily form a basis for $\mathbb{R}^n$

Pf: A has n-columns that are linearly independent. Since $\dim(\mathbb{R}^n) = n$, the columns of A must form a basis.

$\mathcal{S}_n = \{e_1, \ldots, e_n\}$

(104) Does $\mathcal{B} = \left\{ \overset{u^1}{\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}}, \overset{u^2}{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}, \overset{u^3}{\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}} \right\}$ form a basis for $\mathbb{R}^3$?

U

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -6 & -7 & -3 & 1 & 0 \\ 0 & 5 & 7 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 7 & -3 & 5 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -1 & 2 & 2 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -3/7 & \frac{5}{7} & \frac{6}{7} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/7 & \frac{-6}{7} & \frac{-10}{7} \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -3/7 & 5/7 & 6/7 \end{array}\right]$$

$$U^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -6 & -10 \\ 7 & -7 & -7 \\ -3 & 5 & 6 \end{bmatrix}$$

Given $y \in \mathbb{R}^3$ find $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$y = x_1 u^1 + x_2 u^2 + x_3 u^3$$

$$= \left[u^1 \; u^2 \; u^3\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= U x$$

$$\Rightarrow \quad x = U^{-1} y$$

We call the components of x the coordinates of y in the basis $\mathcal{B}$

(105) ex: what are the coordinates of

$[Y]_B = Y$ $\quad Y = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ in the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right\}$

$$x = U^{-1} y = \frac{1}{7} \begin{bmatrix} 5 & -6 & -10 \\ 7 & -7 & -7 \\ -3 & 5 & 6 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{7} \begin{bmatrix} 10 & +6 & +10 \\ 14 & +7 & +7 \\ -6 & -5 & -6 \end{bmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 26 \\ 28 \\ -17 \end{pmatrix} = \begin{pmatrix} 26/7 \\ 4 \\ -17/7 \end{pmatrix}$$

Notation: $[Y]_{\mathcal{B}} = \begin{bmatrix} 26/7 \\ 4 \\ -17/7 \end{bmatrix} = \bar{U}^{-1} Y$, $\quad \bar{U} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 5 \\ -2 & 1 & -1 \end{bmatrix}$

$\Downarrow$

the vector that gives y in the $\mathcal{B}$ coordinates.

Conversion: $[Y]_{\{e_1, e_2, \dots, e_n\}} = Y$