(A) 110521 Review

<u>Theorem</u>: If $S \subset \mathbb{R}^n$ is a subspace, then every basis of S has the same number of elements.

<u>Def</u>: The number of vectors in any basis for a subspace S is called the dimension of S and written $\dim(S)$.

<u>Theorem</u>: Let $S \subset \mathbb{R}^n$ be a subspace with $\dim(S) \geq 1$.
Let $\{u^1, u^2, \dots, u^k\} \subset S$.

(i) If $u^1, u^2, \dots, u^k$ are linearly independent. Then $\dim(S) \geq k$.

(ii) If $u^1, u^2, \dots, u^k$ spans S, then $\dim(S) \leq k$.

(B) Do the vectors $\begin{pmatrix}1\\0\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\2\end{pmatrix}$ form a basis for

$$S' = \operatorname{Span}\left[\begin{pmatrix}1\\1\\3\\1\end{pmatrix}, \begin{pmatrix}2\\1\\5\\0\end{pmatrix}, \begin{pmatrix}1\\3\\5\\5\end{pmatrix}\right] \quad ?$$

(I)

$$\begin{array}{cccc}
1 & 0 & 2 & -1\\
0 & 1 & 1 & 2\\
1 & 1 & 3 & 1\\
2 & 1 & 5 & 0\\
1 & 3 & 5 & 5\\
\hline
1 & 0 & 2 & -1\\
0 & 1 & 1 & 2\\
0 & 1 & 1 & 2\\
0 & 1 & 1 & 2\\
0 & 3 & 3 & 6\\
\hline
1 & 0 & 2 & -1\\
0 & 1 & 1 & 2\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{array}$$

$$\begin{array}{cc|ccc}
1 & 0 & 1 & 2 & 1\\
0 & 1 & 1 & 1 & 3\\
2 & 1 & 3 & 5 & 5\\
-1 & 2 & 1 & 0 & 5\\
\hline
\end{array}$$

(89) Unifying Theorem : Version 5

$\mathcal{S} = \{a^1, a^2, \ldots, a^n\} \subset \mathbb{R}^n$ $A = [a^1, a^2, \ldots, a^n]$

$T_A(x) = Ax$, $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is linear

TFAE

(a) $\mathcal{S}$ spans $\mathbb{R}^n$

(b) $\mathcal{S}$ is linearly independent

(c) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$

(d) T_A is onto

(e) T_A is one-to-one

(f) A is invertible

(g) $\mathrm{Nul}(A) = \{0\}$

(h) $\ker(T_A) = \{0\}$

(i) $\mathcal{S}$ is a basis for $\mathbb{R}^n$

(90) ex: Find a basis for

$$\text{span}\left[\begin{pmatrix}1\\-1\\0\\2\end{pmatrix},\begin{pmatrix}2\\-5\\9\\7\end{pmatrix},\begin{pmatrix}0\\1\\-3\\-1\end{pmatrix}\right]$$

$$\begin{array}{ccc}1 & 2 & 0\\ -1 & -5 & 1\\ 0 & 9 & -3\\ 2 & 7 & -1\end{array}$$

$$\begin{array}{ccc}1 & 2 & 0\\ 0 & -3 & 1\\ 0 & 9 & -3\\ 0 & 1 & -1\end{array}$$

$$\begin{array}{ccc}1 & 2 & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}$$

ex (4) Give an example of two subspaces $S_1 \subset S_2 \subset \mathbb{R}^5$ such that $\dim(S_1)+2=\dim(S_2)$

$$S_1 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}$$

$$S_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\right\}$$

(ii) Give an example of two two dimensional subspaces S_1 and S_2 in $\mathbb{R}^4$ such that $\dim(S_1 \cap S_2) = 1$

$$S_1 = \text{span}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$S_2 = \left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$$

is $S_1 \cap S_2$ a subspace?

(91) Row and Column Spaces of a Matrix

Let $A \in \mathbb{R}^{n \times m}$ with

$$A = [u^1 \ u^2 \ \cdots \ u^m] \quad , u^i \in \mathbb{R}^n \quad i = 1 \ldots m$$

$$= \begin{bmatrix} (v^1)^T \\ (v^2)^T \\ \vdots \\ (v^n)^T \end{bmatrix} \quad , v^j \in \mathbb{R}^m \quad j = 1 \ldots n$$

The row space of A is $\text{span}[v^1, v^2, \ldots, v^n] = \text{row}(A)$
$= \text{Ran}[A^T]$

The column space of A is $\text{span}[u^1, u^2, \ldots, u^m]$
$= \text{col}(A) = \text{Ran}(A)$

(92) What are the row and column spaces for

$$A = \begin{bmatrix} 1 & 0 & -4 & -3 \\ -2 & 1 & 13 & 5 \\ 0 & 1 & 5 & -1 \end{bmatrix} \quad ?$$

Give bases for both row(A) and col(A) as well as their dimensions

$$\begin{matrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 5 & -1 \\ 0 & 1 & 5 & -1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

col sp basis is

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

row sp basis

$$\begin{pmatrix} 1 \\ 0 \\ -4 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix}$$

$$\text{Ran}(A) = \text{span}\left[\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right], \quad \dim(\text{Ran}(A)) = 2$$

$$\text{Ran}(A^T) = \text{span}\left[\begin{pmatrix} 1 \\ 0 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \\ -1 \end{pmatrix}\right], \quad \dim(\text{Ran}(A^T)) = 2$$

(93) Theorem: Let $A \in \mathbb{R}^{n \times m}$ have echelon form B.

(a) The nonzero rows of B form a basis for $\mathrm{row}(A)$.

(b) The columns of A associated with the pivot columns in B form a basis for $\mathrm{col}(A)$.

(c) $\dim(\mathrm{row}(A)) = \dim(\mathrm{col}(A))$

$[\dim(\mathrm{Ran}(A)) \stackrel{!}{=} \dim(\mathrm{Ran}(A^T))]$

$\mathrm{row}(A) \subset \mathbb{R}^m \qquad \mathrm{col}(A) \subset \mathbb{R}^n$

Definition: The rank of a matrix $A \in \mathbb{R}^{n \times m}$ is the dimension of the column space of A which equals the dimension of the row space of A

$$\mathrm{rank}(A) = \dim(\mathrm{col}(A))$$

(94) <u>Thm</u>: The rank plus nullity Theorem

Given $A \in \mathbb{R}^{n \times m}$ we have

$$\text{rank}(A) + \text{nullity}(A) = m$$

where $\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{row}(A))$

$\text{nullity}(A) = \dim(\text{Nul}(A))$.

<u>pf</u>: Let $B \in \mathbb{R}^{n \times m}$ be the echelon form of A

Then the number of non-zero rows of B equals $\text{rank}(A)$ which equals the number of pivots in B.

The nullity of A equals the number of free variables in B which equals $n -$ the number of pivots $= n - \text{rank}(A)$.

Therefore, $\text{nullity}(A) = n - \text{rank}(A)$.

(95) Find the rank and nullity of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix} \in \mathbb{R}^{3\times 5}$$

$$\begin{matrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & -1 & 3 & -5 \end{matrix}$$

$$\begin{matrix} 1 & -2 & 3 & 0 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

rank(A) = 2

basis for row(A) = $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 9 \\ -16 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 5 \end{pmatrix} \right\}$

basis for col(A) = $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix} \right\}$

nullity(A) = 3 (3 free variables)

$5 = n = \text{rank}(A) + \text{nullity}(A) = 2 + 3$

Basis for Nul(A) = $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Check

(96) Find a linear transformation T that has

$$\ker(T) = \text{span}\left[\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\\2\end{pmatrix}\right]$$

Note $\begin{bmatrix}1 & 0\\0 & 1\\-2 & 3\\1 & 2\end{bmatrix} = \begin{bmatrix}I\\M\end{bmatrix}$, $[-M, I]\begin{bmatrix}I\\M\end{bmatrix} = -M+M=0$

$$[-M \;\; I] = \begin{bmatrix}2 & -3 & 1 & 0\\-1 & -2 & 0 & 1\end{bmatrix} = [-M, I]$$

Set $T = T_{[-M\, I]}$. Then $\ker(T) = \text{span}\left[\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\3\\2\end{pmatrix}\right]$

Find A so that

$$[0, 0] = \left[A\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}, A\begin{pmatrix}0\\1\\3\\2\end{pmatrix}\right] = A\begin{bmatrix}1 & 0\\ 0 & 1\\ \hline -2 & 3\\ 1 & 2\end{bmatrix} = A\begin{bmatrix} I \\ M\end{bmatrix}$$

$$A = [-M \ \ I] \Rightarrow A\begin{bmatrix} I \\ M\end{bmatrix} = [-M, I]\begin{bmatrix} I \\ M\end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -3 & 1 & 0\\ -1 & -2 & 0 & 1\end{bmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ -2 & 3\\ 1 & 2\end{pmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

ker $\quad T = T_A \qquad A =$

$$\ker(T_A) = \mathrm{Nul}(A)$$

(97) Find a linear transformation T that has

$$\ker(T) = \operatorname{span}\left[\begin{pmatrix} -1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 10 \\ 2 \\ 1 \end{pmatrix} \right]$$