(A) Find a basis for $\operatorname{Span}\left[\begin{pmatrix}1\\-2\\3\\-2\end{pmatrix}, \begin{pmatrix}0\\2\\-5\\1\end{pmatrix}, \begin{pmatrix}2\\-2\\1\\-3\end{pmatrix}\right]$

Method 1

$$\left.\begin{array}{c}\begin{matrix}1 & -2 & 3 & -2\\ 0 & 2 & -5 & 1\\ 2 & -2 & 1 & -3\end{matrix}\\ \hline \begin{matrix}1 & -2 & 3 & -2\\ 0 & 2 & -5 & 1\\ 0 & 2 & -5 & 1\end{matrix}\\ \hline \begin{matrix}1 & -2 & 3 & -2\\ 0 & 2 & -5 & 1\\ 0 & 0 & 0 & 0\end{matrix}\end{array}\right\} \overset{\text{Basis}}{\Longrightarrow} \begin{pmatrix}1\\-2\\3\\-2\end{pmatrix}, \begin{pmatrix}0\\2\\-5\\1\end{pmatrix}$$

2^{nd} Method: Recall that the matrices $A, B \in \mathbb{R}^{n\times m}$ are equivalent if and only if there is an invertible matrix $G \in \mathbb{R}^{n\times n}$ such that

$$A = GB$$

This implies that every dependence relationship between the columns of B is a dependence relationship between the columns of A, i.e. $Ax = 0 \Leftrightarrow Bx = 0$

(B) Consequently, if

$$A = [a^1\ a^2 \dots\ a^m]$$
$$B = [b^1\ b^2 \dots\ b^m]$$

then (i) $a^{l_1}, a^{l_2}, \dots, a^{l_k}$ are linearly dependent

$$\Longleftrightarrow$$

$b^{l_1}, b^{l_2}, \dots, b^{l_k}$ are linearly dependent

(ii) $a^{l_1}, a^{l_2}, \dots, a^{l_k}$ are linearly independent

$$\Longleftrightarrow$$

$b^{l_1}, b^{l_2}, \dots, b^{l_k}$ are linearly independent

(c)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 2 & -2 \\ 3 & -5 & 1 \\ -2 & 1 & -3 \end{bmatrix}$$

$$\begin{matrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & -5 & -5 \\ 0 & 1 & 1 \end{matrix}$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

linearly independant

The pivot columns of B are linearly independant so the corresponding columns of A are linearly independant!

Basis for column space of A is

$$\begin{pmatrix} 1 \\ -2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -5 \\ 1 \end{pmatrix}$$

(85) 2^{nd} Method for Finding a basis for the span of $u^1, u^2, \ldots, u^k$.

Step 1: $A = [u^1\ u^2 \cdots u^k] \xRightarrow[\text{form}]{\text{echelon}} [v^1\ v^2 \ldots v^k] = V$

Step 2: The pivot columns of V give you the columns of A that form a basis for $\{u^1, u^2, \ldots, u^k\}$, i.e. the pivot colum

ex: $u^1 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, u^2 = \begin{pmatrix} -6 \\ 7 \\ 5 \\ 2 \end{pmatrix}$ form a basis for

$S = \text{span}\left[\begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 7 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}\right]$

$\text{span}\{u^1, u^2, u^3\}$

$= \text{span}\{u^1, u^2, u^3\}$

(86) Dimension

Theorem: If S is a subspace in $\mathbb{R}^n$, then every basis of S has the same number of vectors.

Definition: Let S be a subspace of $\mathbb{R}^n$. Then The dimension of S is the number of vectors in any basis for S.

notation: $\dim(S)$ = dimension of S.

86.5 <u>Thm</u>: If S is a subspace of $\mathbb{R}^n$, Then every basis of S has the same number of elements.

<u>Pf</u>: Suppose $\{u^1, u^2, \ldots, u^k\}$ and $\{v^1, v^2, \ldots, v^\ell\}$ are two bases for S with $k < \ell$.

set $\bar{U} = [u^1\ u^2 \ldots u^k] \in \mathbb{R}^{n \times k}$ and $\bar{V} = [v^1 v^2 \ldots v^\ell] \in \mathbb{R}^{n \times \ell}$

$\mathrm{Ran}[\bar{U}] = \mathrm{Ran}[\bar{V}] = S$

$v^i = \bar{U} x^i, \ x^i \in \mathbb{R}^k \quad i = 1, 2, \ldots \ell$ $\qquad \bar{X} =$

$\Rightarrow \bar{V} = [\bar{U} x^1, \bar{U} x^2, \ldots, \bar{U} x^\ell] = \bar{U} \bar{X}$

$\bar{X} = [x^1\ x^2 \ldots x^\ell] \in \mathbb{R}^{k \times \ell}$

$\Rightarrow 0 = \bar{X} y \Rightarrow 0 = \bar{U} \bar{X} y \Leftrightarrow \bar{V} y$

But $k < \ell$ so $\exists\ y \in \mathbb{R}^\ell$ with $y \neq 0$ s.t. $\bar{X} y = 0$

so $\bar{V} y = 0$ a contradiction so $k = \ell$.

(87)

Thm: Let $S \subset \mathbb{R}^n$ be a subspace with $\dim(S) \geq 1$.

Let $\{u^1, \ldots, u^k\} \subset S$.

(a) If $\{u^1, \ldots, u^k\}$ is linearly independent then
$$\dim(S) \geq k.$$

(b) If $\{u^1, u^2, \ldots, u^k\}$ span S then $\dim(S) \leq k$.

ex: Expand the vectors $\left\{ \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$ to a basis for $\mathbb{R}^3$.

(i) $A = \begin{bmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{matrix} 1 & -2 & -1 & 0 & 0 \\ 0 & 6 & 2 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & -2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -4 & 1 & 6 \end{matrix}$$

$\Rightarrow$ $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ form a basis for $\mathbb{R}^3$

98

Let $\mathcal{U} = \{u^1, u^2, \dots, u^k\}$ be k vectors in a subspace S of dimension k.
If $\mathcal{U}$ is either linearly independent or spans S, then $\mathcal{U}$ is a basis for S.

ex: $\mathcal{U} = \left\{ \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Does $\mathcal{U}$ form a basis for $\operatorname{span}\left[\begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right]$?

(89) Unifying Theorem : Version 5

$\mathcal{S} = \{a^1, a^2, \dots, a^n\} \subset \mathbb{R}^n$ $\quad A = [a^1, a^2, \dots, a^n]$

$T_A(x) = Ax$, $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is linear

TFAE

(a) $\mathcal{S}$ spans $\mathbb{R}^n$

(b) $\mathcal{S}$ is linearly independent

(c) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$

(d) T_A is onto

(e) T_A is one-to-one

(f) A is invertible

(g) $\mathrm{Nul}(A) = \{0\}$

(h) $\ker(T_A) = \{0\}$

(i) $\mathcal{S}$ is a basis for $\mathbb{R}^n$

(90) ex: Find a basis for

$$\text{span}\left[\begin{pmatrix}1\\-1\\0\\2\end{pmatrix},\begin{pmatrix}2\\-5\\9\\7\end{pmatrix},\begin{pmatrix}0\\1\\-3\\-1\end{pmatrix}\right]$$

$$\begin{array}{ccc}1&2&0\\-1&-5&1\\0&9&-3\\2&7&-1\\\hline 1&2&0\\0&-3&1\\0&9&-3\\0&1&-1\\\hline 1&2&0\\0&1&-1\\0&0&0\\0&0&0\end{array}$$

ex. Give an example of two subspaces $S_1 \subset S_2 \subset \mathbb{R}^5$ such that $\dim(S_1)+2=\dim(S_2)$

$$S_1 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}$$

$$S_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\right\}$$

(ii) Give an example of two two dimensional subspaces S_1 and S_2 in $\mathbb{R}^4$ such that $\dim(S_1 \cap S_2)=1$

is $S_1 \cap S_2$ a subspace?

$$S_1 = \text{span}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$S_2 = \left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$$