78 4.2 Basis and Dimension

We now combine the concepts of linear independence and spanning sets.

Defn: A set $\mathcal{B} = \{u^1, u^2, u^3, \dots, u^k\} \subset \mathbb{R}^n$ is a basis for a subspace $S \subset \mathbb{R}^n$ if

(a) $\mathcal{B}$ spans S, i.e. $\text{Span}\, \mathcal{B} = S$, and

(b) $\mathcal{B}$ is linearly independent.

ex (i) Give a basis for $\mathbb{R}^2$ $\quad \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

(ii) Give a basis for $\mathbb{R}^4$ $\quad \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$

(iii) Give a basis for $\mathbb{R}^n$

79 <u>Theorem</u>: If $\mathcal{B} = \{u^1, u^2, \dots, u^k\}$ is a basis for the subspace S, then for every $s \in S$ there is a unique set of coefficients $\sigma_1, \sigma_2, \dots, \sigma_k \in \mathbb{R}$ such that

$$s = \sigma_1 u^1 + \sigma_2 u^2 + \dots + \sigma_k u^k$$

<u>Pf</u>: Suppose $\exists \begin{matrix} \sigma_1 & \sigma_2 & \dots & \sigma_k \\ \hat{\sigma}_1 & \hat{\sigma}_2 & \dots & \hat{\sigma}_k \end{matrix} \subset \mathbb{R}^k$

$$s = \sigma_1 u^1 + \sigma_2 u^2 + \dots + \sigma_k u^k$$
$$s = \hat{\sigma}_1 u^1 + \hat{\sigma}_2 u^2 + \dots + \hat{\sigma}_k u^k$$

subtract $0 = (\sigma_1 - \hat{\sigma}_1) u^1 + (\sigma_2 - \hat{\sigma}_2) u^2 + \dots + (\sigma_k - \hat{\sigma}_k) u^k$

But $u^1, u^2 \dots u^k$ linearly independent

$\Rightarrow \sigma_1 = \hat{\sigma}_1 \quad \sigma_2 = \hat{\sigma}_2 \dots \quad \sigma_k = \hat{\sigma}_k$

so $\sigma_1, \sigma_2, \dots, \sigma_k$ are unique

(80) Let $A, B \in \mathbb{R}^{n \times m}$

Recall A is equivalent to B if for all $b \in \mathbb{R}^n$

$$\{x : Ax = b\} = \{x : Bx = b\}$$

$\iff$ A can be transformed into B by a sequence of elementary row operations (and vis-a-versa)

$\iff$ [the span of the rows of A]

$=$

[the span of the rows of B]

$\iff$ $\exists$ an invertible matrix G s.t.

$$A = GB$$

(81) Find a basis for the span of the vectors $u^1 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$, $u^2 = \begin{pmatrix} -6 \\ 7 \\ 5 \\ 2 \end{pmatrix}$, $u^3 = \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{matrix} -1 & 2 & 3 & 1 \\ -6 & 7 & 5 & 2 \\ 4 & -3 & 1 & 0 \end{matrix}$$

$$\begin{matrix} -1 & 2 & 3 & 1 \\ 0 & -5 & -13 & -4 \\ 0 & 5 & 13 & 4 \end{matrix}$$

$$\begin{matrix} -1 & 2 & 3 & 1 \\ 0 & 5 & 13 & 4 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 13 \\ 4 \end{pmatrix}$$

(82) Steps to finding a basis for Span$[u^1, u^2, u^3, \ldots u^k]$

Step 1: Form the matrix A whose rows are $u^1, u^2, \ldots, u^k$

$$A = \begin{bmatrix} (u^1)^T \\ (u^2)^T \\ \vdots \\ (u^k)^T \end{bmatrix}$$

Step 2: Reduce A to echelon form

$$B = \begin{bmatrix} (w^1)^T \\ (w^2)^T \\ \\ (w^k)^T \end{bmatrix}$$

Step 3: A basis for Span$[u^1, u^2, \ldots u^k]$ is given by the non-zero rows of B

$$\{ w^i : w^i \neq 0, \ i = 1, 2, \ldots, k \}$$

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$$\begin{array}{cccc} -1 & 2 & 3 & 1 \\ -6 & 7 & 5 & 2 \\ 4 & -3 & 1 & 0 \end{array}$$

$$\begin{array}{cccc} 1 & -2 & -3 & -1 \\ 0 & -5 & -13 & -4 \\ 0 & 5 & 13 & 4 \end{array}$$

$$\begin{array}{cccc} 1 & -2 & -3 & -1 \\ 0 & 5 & 13 & 4 \\ 0 & 0 & 0 & 0 \end{array}$$

so a basis for $\text{Span}\left\{ \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 7 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$

is $\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 13 \\ 4 \end{pmatrix} \right\}$

(83) <u>Theorem</u>: Suppose

$\bar{U} = [u^1\ u^2 \cdots u^m] \in \mathbb{R}^{n\times m}$ and $\bar{V} = [v^1\ v^2 \cdots v^m] \in \mathbb{R}^{n\times m}$

are equivalent matrices. Then <u>$Nul(\bar{U}) = Nul(\bar{V})$</u>

and

(i) [$u^1, u^2, \ldots, u^m$ are linearly independent]

$$\iff$$

[$v^1, v^2, \ldots, v^m$] are linearly independent]

(ii) $u^1, u^2, \ldots, u^m$ are linearly dependent

$$\iff$$

$v^1, v^2, \ldots, v^m$ are linearly dependent

(iii) Any dependence relationship between the columns of $\bar{U}$ is also a dependence relationship between the columns of $\bar{V}$!

(84) $\bar{U} = [u^1 \; u^2 \; u^3]$

$$= \begin{bmatrix} -1 & -6 & 4 \\ 2 & 7 & -3 \\ 3 & 5 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\begin{matrix} 1 & 6 & -4 \\ 0 & -5 & 5 \\ 0 & -13 & 13 \\ 0 & -4 & 4 \end{matrix}$$

$$V = \begin{bmatrix} 1 & 6 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= [v^1 \; v^2 \; v^3]$

Observe

$$2v^1 - v^2 = v^3$$

Consequently,

$$2u^1 - u^2 = u^3 \; !$$

Why?

$E\bar{U} = \bar{V}$ and E^{-1} exists

(LU Factorization)

So if $0 = Vx = EUx$

$\Leftrightarrow \; 0 = EUx$

$\Leftrightarrow \; 0 = \bar{U}x$

Therefore, $\left\{ \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 7 \\ 5 \\ 2 \end{pmatrix} \right\}$

form a basis for

span $[u^1, u^2, u^3]$