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$$M = \begin{array}{c c} & \begin{array}{cc} 3 & \ell \end{array} \\ \begin{array}{c} k \\ a \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{array}{c} \}2 \\ \}5 \end{array} \\ & \begin{array}{cc} b & 4 \end{array} \end{array} \in \mathbb{R}^{n \times m}$$

$A_{11} \in \mathbb{R}^{k \times 3}$ $\quad A_{12} \in \mathbb{R}^{2 \times \ell}$ $\quad A_{21} \in \mathbb{R}^{a \times b}$ $\quad A_{22} \in \mathbb{R}^{5 \times 4}$

what are k, ℓ, a, b, m and n?

$k = 2$, $\ell = 4$, $a = 5$, $b = 3$, $m = 3 + \ell = 3 + 4 = 7$

$n = 2 + 5 = 7$

(70) Gauss - Jordan Elimination Matrices

$$x = \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} \qquad \begin{matrix} a \in \mathbb{R}^k \\ \alpha \in \mathbb{R}^1 \quad \alpha \neq 0 \\ b \in \mathbb{R}^s \end{matrix} \quad \Rightarrow \quad x \in \mathbb{R}^{k+1+s}, \quad n = k+1+s$$

Find a matrix $G \in \mathbb{R}^{n \times n}$ such that $Gx = \begin{pmatrix} 0_k \\ 1 \\ 0_s \end{pmatrix}$.

$$G = \begin{bmatrix} I_{k \times k} & -\alpha^{-1} a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1} b & I_{s \times s} \end{bmatrix} \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 1 \\ 1 \end{pmatrix} \rightarrow \alpha = 4$$

$$G^{-1} = \begin{bmatrix} I_{k \times k} & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I_{s \times s} \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 1 & 0 \\ 0 & 0 & -1/4 & 0 & 1 \end{bmatrix}$$

71 Subspaces

<u>Defn</u>: A subset $S \subset \mathbb{R}^n$ is called a subspace if S satisfies the following 3 conditions:

(a) $0_n \in S$

(b) If $u, v \in S$, then $u + v \in S$. [closed under addition]

(c) If $\alpha \in \mathbb{R}$ and $u \in S$, Then $\alpha u \in S$ [closed under scalar multiplication]

<u>ex</u>: (i) Is $\operatorname{span}\left[\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}4\\0\\2\end{pmatrix}\right]$ a subspace?

(ii) $u, v \in \mathbb{R}^n$. Is $\operatorname{Span}[u, v]$ a subspace?

(iii) $u^1, u^2, \ldots, u^k \in \mathbb{R}^n$ Is $\operatorname{Span}[u^1, u^2, \ldots, u^k]$ a subspace?

(iv) $A \in \mathbb{R}^{n \times m}$ Is $\operatorname{Ran}(A)$ a subspace?

(72) Let $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$.

When is the set of all solutions to

$$Ax = b$$

a subspace?

Is $\{ y : \exists\, x \in \mathbb{R}^m \text{ such that } y = Ax \}$

a subspace?

73 Steps for Checking if a set $S \subset \mathbb{R}^n$ is a subspace.

Step 1: Check if $0 \in S$. If not, S is not a subspace.

Step 2: If $S = \text{Span}\{u^1, \ldots, u^k\}$, then S is a subspace

Step 3: (i) If $u, v \in S$, is $u+v \in S$? If not, S is not a subspace.

(ii) If $u \in S$, $\alpha \in \mathbb{R}$, is $\alpha u \in S$? If not, S is not a subspace

(iii) If (a) $0 \in S$, (b) $u+v \in S$ $\forall u, v \in S$, (c) $\alpha u \in S$ $\forall u \in S, \alpha \in \mathbb{R}$ then S is a subspace.

$$A \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

ex: $S \subset \mathbb{R}^7$, $\forall u \in S$ $u_4 = u_5 = 0$.

Is S a subspace?

$\text{Null}(A) = S$

<u>Theorem</u>: If $A \in \mathbb{R}^{n \times m}$, then

$$\mathrm{Nul}(A) = \{ x \in \mathbb{R}^m : Ax = 0 \}$$

is a subspace.

<u>ex</u>:

$$S = \left\{ x \in \mathbb{R}^6 : \begin{array}{l} x_1 + x_2 = x_3 + x_5 \\ x_6 = x_3 - x_2 \end{array} \right\}$$

is S a subspace?

$$S = \mathrm{Nul}\left(\begin{bmatrix} 1 & 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \end{bmatrix} \right)$$

(75)

- $T : \mathbb{R}^m \to \mathbb{R}^n$ a linear transformation $\quad \ker(T) = \{x \in \mathbb{R}^m : T(x) = 0\}$

- $A \in \mathbb{R}^{m \times n}$, $T_A(x) = Ax$, $\ker(T_A) = \mathrm{Nul}(A)$.

- $\mathrm{Ran}(T) = \{y : \exists x \in \mathbb{R}^m \text{ s.t. } T(x) = y\} \subset \mathbb{R}^n$ is a subspace

- $\mathrm{Ran}(T_A) = \mathrm{Ran}(A)$

- $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation

  T is one-to-one $\Leftrightarrow$ T is onto

(76) $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation

Suppose $T(e_1) = a^1$, $T(e_2) = a^2$, ..., $T(e_m) = a^m$

Set $A = [a^1 \; a^2 \; \cdots \; a^m]$

Then $T(x) = Ax = T_A(x)$

<u>Unifying Theorem: Version 4</u>

$S = \{u^1, u^2, \ldots, u^n\} \subset \mathbb{R}^n$, $A = [u^1 \; u^2 \cdots u^n]$, $T_A(x) = Ax$

TFAE

(a) $\operatorname{Span} S = \mathbb{R}^n$

(b) S is linearly independent

(c) $Ax = b$ has a unique solution $\forall b \in \mathbb{R}^n$

(d) T_A is onto

(e) T_A is one-to-one

(f) A is invertible

(g) $\ker(T_A) = \{0\}$

(h) $T_A^{-1} = T_{A^{-1}}$

(77) Find the nullspace for the matrix

$$\begin{bmatrix} -1 & 0 & 0 & 4 & 5 & 2 \\ 6 & 2 & 1 & 2 & 4 & 0 \\ 3 & 2 & -5 & 2 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} I & T \end{bmatrix} \begin{bmatrix} -T \\ I \end{bmatrix} = -T + T$$

$$= 0$$

$$\begin{matrix} 1 & 0 & 0 & -4 & -5 & -2 \\ 0 & 2 & 1 & 26 & 34 & 12 \\ 0 & 2 & -5 & 14 & 16 & 6 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & -4 & -5 & -2 \\ 0 & 2 & 1 & 26 & 34 & 12 \\ 0 & 0 & -6 & -12 & -18 & -6 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & -4 & -5 & -2 \\ 0 & 2 & 0 & 24 & 31 & 11 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & -4 & -5 & -2 \\ 0 & 1 & 0 & 12 & 31/2 & 11/2 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{matrix}$$

$I \qquad T$

$$\begin{pmatrix} 4 & 5 & 2 \\ -12 & -31/2 & -11/2 \\ -2 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$