Watch video:

Ch. 3 : Essence of Linear Algebra

Just type "Essence of Linear Algebra"

into your browser to find it.

(62) Theorem: Let $A, B \in \mathbb{R}^{n \times n}$ be invertible and let $C, D \in \mathbb{R}^{n \times m}$.

(a) A^{-1} is invertible with $(A^{-1})^{-1} = A$

(b) AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$ (order reversing)

(c) If $AC = AD$, then $C = D$

(d) If $AC = 0_{nm}$, Then $C = 0$

pf (a) $I = (A^{-1})B \Rightarrow B = (A^{-1})^{-1}$ is unique but $I = A^{-1}A$ so $B = A = (A^{-1})^{-1}$

(b) $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$

(c) $AC = AD \Rightarrow A^{-1}(AC) = A^{-1}(AD) \Leftrightarrow C = D$

(d) $A0 = 0 = AC \Leftrightarrow C = 0$

63 Computing $A^{-1} \in \mathbb{R}^{n\times n}$:

Suppose $A^{-1} = B = [b^1\ b^2 \cdots b^n]$

$\Rightarrow [e_1\ e_2 \cdots e_n] = I = AB = [Ab^1\ Ab^2 \cdots Ab^n]$

$\Leftrightarrow \quad Ab^i = e_i \quad \sim i^{th}$ unit coordinate vector

so we only need to solve $Ab^i = e_i$ to get the i^{th} column of A^{-1}

solve all at once

$$[A \,|\, I] \longrightarrow [I \,|\, B] \Rightarrow B = A^{-1}$$

$$[\underbrace{E_k \cdots E_3 E_2 E_1 A} \,|\, E_k \cdots E_3 E_2 E_1]$$

$$[I \qquad | \quad A^{-1}]$$

64

A

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 0 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & -2 & 0 & 6 & 1 & -1 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 1 & 0 & 0 & -18 & -3 & 5 \\ 0 & 1 & 0 & -12 & -2 & 3 \\ 0 & 0 & 1 & -5 & -1 & 1 \end{array}$$

A^{-1}

Check

(65) A simple formula for 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>ex</u>: $A = \begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{2-15}\begin{bmatrix} -1 & -3 \\ -5 & -2 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} -1 & -3 \\ -5 & -2 \end{bmatrix}$$

(66) Unifying Theorem Version 3

$S = \{a^1, a^2, \ldots, a^n\} \subset \mathbb{R}^n$. $A = [a^1\ a^2 \cdots a^n] \in \mathbb{R}^{n \times n}$

$$T_A(x) = Ax$$

TFAE

(a) S spans $\mathbb{R}^n$

(b) S is linearly independent

(c) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$

(d) T is onto

(e) T is one-to-one

(f) T is invertible

(67) Partioned Matrices

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & 0 & 0 \\ 0 & 2 & 7 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad A_{11} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 7 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & A_\ell \end{bmatrix}, \qquad A_i \in \mathbb{R}^{k_i \times k_i} \qquad k_1 + k_2 + \cdots + k_\ell = n$$

A is invertible $\Longleftrightarrow$ A_i is invertible $i = 1, \ldots, \ell$

$$A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & A_\ell^{-1} \end{bmatrix}$$

ex: $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ 0 & & \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ 0 & & \end{bmatrix}$

(68)

$$A = \left[\begin{array}{cc|cc} 3 & -4 & 1 & 1 \\ 2 & 2 & 0 & 1 \\ -2 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$A_{11} = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ -2 & 0 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_{21} = 0_{2\times 2}$$

$$A_{22} = I_{2\times 2}$$

$$B = \left[\begin{array}{cc} 2 & 1 \\ 1 & 0 \\ \hline 1 & 2 \\ 1 & -1 \end{array}\right] \qquad B_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

AB

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$