(54) Explorations

unit coordinate vectors in $\mathbb{R}^m$

(i) $A = [a^1 \ a^2 \cdots a^m] \in \mathbb{R}^{m \times n}$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ position

$e_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$Ae_1 =$

$Ae_2 =$

$Ae_i =$

$Ae_n =$

(53) Diagonal matrices

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix} \in \mathbb{R}^n, \quad \begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix} \in \mathbb{R}^m,$$

$A \in \mathbb{R}^{n\times m}$

$A = [a^1 \ a^2 \ \cdots \ a^m]$ columns

$= \begin{bmatrix}(v^1)^T\\ (v^2)^T\\ \vdots\\ (v^n)^T\end{bmatrix}$ rows

$A^T = [v^1 \ v^2 \ \cdots \ v^n]$

$$\left.\begin{aligned} D_\alpha &= \begin{bmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & 0 \cdots & 0\\ \vdots & & & \\ 0 & \cdots & & \alpha_n\end{bmatrix}\\ D_\beta &= \begin{bmatrix}\beta_1 & 0 & \cdots & 0\\ 0 & \beta_2 & \cdots & 0\\ 0 & \cdots & & \beta_m\end{bmatrix}\end{aligned}\right\}\text{Diagonal matrices}$$

$$D_\alpha A = \begin{bmatrix}\alpha_1 (v^1)^T\\ \alpha_2 (v^2)^T\\ \vdots\\ \alpha_n (v^n)^T\end{bmatrix}$$ re-scaling the rows of A

$$A D_\beta = [\beta_1 a^1 \ \ \beta_2 a^2 \ \cdots \ \beta_m a^m]$$ re-scaling the columns of A

## Elementary Matrices

If we perform a single elementary row operation on an identity matrix, then the resulting matrix is called an "elementary matrix".

ex:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

$$E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

(57)

$$R_1 \leftrightarrow R_3 \quad \overset{E_1}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{bmatrix}} = \begin{bmatrix} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{bmatrix}$$

$$R_3 + 2R_2 \quad \overset{E_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 2 & -2 & -6 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & -1 & -3 \end{bmatrix}$$

$$R_2 + R_1 \quad \overset{E_3}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -2 & -1 & 1 & 2 \\ -1 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & -3 \end{bmatrix}$$

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$$E_3 E_2 E_3 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & -6 & 1 & 3 \\ -1 & 1 & 3 & -1 & -3 \\ 1 & -2 & -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & 1 & 2 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 & -3 \end{bmatrix}$$

Echelon Form can be obtained by successive multiplication by elementary matrices. Similarly, Gauss-Jordan Form can be obtained in the same way.

58 Sec. 3.3 Matrix Inverses

solve
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= y_1 \\ -2x_1 - 7x_2 + 2x_3 &= y_2 \\ 2x_1 + 6x_2 + 5x_3 &= y_3 \end{aligned}, \quad Ax = y$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & y_1 \\ -2 & -7 & 2 & y_2 \\ 2 & 6 & 5 & y_3 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & y_1 \\ 0 & -1 & 6 & 2y_1 + y_2 \\ 0 & 0 & 1 & -2y_1 + y_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 20 & 7y_1 + 3y_2 \\ 0 & -1 & 0 & 14y_1 + y_2 - 6y_3 \\ 0 & 0 & 1 & -2y_1 + y_3 \end{array}\right]$$

$$\begin{array}{l} R_1 - 20R_3 \\ -R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 47y_1 + 3y_2 - 20y_3 \\ 0 & 1 & 0 & -14y_1 - y_2 + 6y_3 \\ 0 & 0 & 1 & -2y_1 + y_3 \end{array}\right]$$

$$x = \begin{bmatrix} 47 & 3 & -20 \\ -14 & -1 & 6 \\ -2 & 0 & 1 \end{bmatrix} y = By$$

$$x = By$$

$$Ax = y, \quad By = x$$

$$\begin{array}{l|l} A(By) = y & B(Ax) = x \\ (AB)y = y & (BA)x = x \end{array}$$

$$AB = BA = I$$

(59) $AB = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -7 & 2 \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 47 & 3 & -20 \\ -14 & -1 & 6 \\ -2 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$BA = \begin{bmatrix} 47 & 3 & -20 \\ -14 & -1 & 6 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ -2 & -7 & 2 \\ 2 & 6 & 5 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We call $B$ the inverse of $A$ and write $B = A^{-1}$. since $AB = BA = I$

Note: $(B^{-1})^{-1} = A$

$T_A(T_B(y)) = ABy = y$

(60) The linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is invertible if $T$ is one-to-one and onto (in particular, this implies that $m = n$).

When $T$ is invertible, the inverse function $T^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$T^{-1}(y) = x \iff T(x) = y.$$

<u>Fact</u>: $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.

(a) If $T$ is invertible, then $m = n$.

(b) If $T$ is invertible, then $T^{-1}$ is also a linear transformation.

(G1) We say $A \in \mathbb{R}^{n \times n}$ is invertible if there is a matrix $B \in \mathbb{R}^{n \times n}$ such that $AB = I_{n \times n}$ and write $B = A^{-1}$.

Fact: If $B$ is an inverse of $A \in \mathbb{R}^{n \times n}$, Then $BA = I$ and $B$ is unique.

Pf: $AB = I_n$ so $AB(Ax) = Ax \Rightarrow A(BAx) = Ax$

$\Rightarrow A(BAx - x) = 0$

$A$ one-to-one $\Rightarrow BAx = x \quad \forall x$

$\Rightarrow BA = I_n$

If $AB = I = AC$, then

$B = B(AB) = B(AC) = (BA)C = C$

(62) Theorem: Let $A, B \in \mathbb{R}^{n \times n}$ be invertible and let $C, D \in \mathbb{R}^{n \times m}$.

(a) $A^{-1}$ is invertible with $(A^{-1})^{-1} = A$

(b) $AB$ is invertible with $(AB)^{-1} = B^{-1}A^{-1}$ (order reversing)

(c) If $AC = AD$, then $C = D$

(d) If $AC = 0_{nm}$, Then $C = 0$

pf (a) $I = (A^{-1})B \Rightarrow B = (A^{-1})^{-1}$ is unique but $I = A^{-1}A$ so $B = A = (A^{-1})^{-1}$

(b) $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$

(c) $AC = AD \Rightarrow A^{-1}(AC) = A^{-1}(AD) \Leftrightarrow C = D$

(d) $A0 = 0 = AC \Leftrightarrow C = 0$

(63) Computing $A^{-1} \in \mathbb{R}^{n\times n}$

Suppose $A^{-1} = B = [b^1 \; b^2 \; \cdots \; b^n]$

$\Rightarrow [e_1 \; e_2 \cdots e_n] = I = AB = [Ab^1 \; Ab^2 \cdots Ab^n]$

$\Leftrightarrow \quad Ab^i = e_i$ ~ $i^{th}$ unit coordinate vector

so we only need to solve $Ab^i = e_i$ to get the $i^{th}$ column of $A^{-1}$

solve all at once

$$[A \mid I] \longrightarrow [I \mid B] \Rightarrow B = A^{-1}$$

(64)

$$
\begin{array}{c}
A \\
\left[\begin{array}{ccc|ccc}
1 & -2 & 1 & 1 & 0 & 0 \\
-3 & 7 & -6 & 0 & 1 & 0 \\
2 & -3 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$$
\left[\begin{array}{ccc|ccc}
1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & -3 & 3 & 1 & 0 \\
0 & 1 & -2 & -2 & 0 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{ccc|ccc}
1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & -3 & 3 & 1 & 0 \\
0 & 0 & 1 & -5 & -1 & 1
\end{array}\right]
$$

$$
\left[\begin{array}{ccc|ccc}
1 & -2 & 0 & 6 & 1 & -1 \\
0 & 1 & 0 & -12 & -2 & 3 \\
0 & 0 & 1 & -5 & -1 & 1
\end{array}\right]
$$

$$
\begin{array}{c}
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & -18 & -3 & 5 \\
0 & 1 & 0 & -12 & -2 & 3 \\
0 & 0 & 1 & -5 & -1 & 1
\end{array}\right] \\
\qquad\qquad\quad A^{-1}
\end{array}
$$

Check