(44) Sec 3.2 Matrix Algebra

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -7 & 4 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 & 8 \\ 10 & -8 & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+2 & -3+5 & 2+8 \\ -7+10 & 4-8 & 3+1 \end{bmatrix}$$ componentwise addition

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \end{bmatrix}$$

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

(45) Matrix scalar multiplication is again componentwise

$$3\begin{bmatrix} 2 & 1 & -1 \\ 4 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 & 3\cdot 1 & 3(-1) \\ 3\cdot 4 & 3\cdot 7 & 3\cdot 9 \end{bmatrix} = \begin{bmatrix} 6 & 3 & -3 \\ 12 & 21 & 27 \end{bmatrix}$$

$$\alpha\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1m} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2m} \\ \vdots & & & \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nm} \end{bmatrix}$$

Properties:

(a) $A + B = B + A$

(b) $\alpha(A+B) = \alpha A + \alpha B$

(c) $(\alpha+\beta)A = \alpha A + \beta A$

(d) $(A+B)+C = A + (B+C) = (A+C) + B$

(e) $(st)A = s(tA)$

(f) $A + O_{nm} = A$

$$O = O_{n\times m} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times m}$$

(46) Matrix Multiplication : $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times k}$

matrix vector

$$Ax = [a^1 \; a^2 \; \cdots \; a^m] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_1 a^1 + x_2 a^2 + \cdots + x_m a^m$$

$$B = [b^1 \; b^2 \; \cdots \; b^k] \quad , \quad b^i \in \mathbb{R}^m \quad i = 1, 2, \ldots k$$

$$AB = A [b^1 \; b^2 \; \cdots \; b^k]$$

$$= [Ab^1 \; Ab^2 \; Ab^3 \; \cdots \; Ab^k] \in \mathbb{R}^{n \times k}$$

ex: $A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ -2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 4 & -1 & 2 & 1 \end{bmatrix}$ $\quad A \in \mathbb{R}^{3 \times 2}$, $B \in \mathbb{R}^{2 \times 4}$

$$AB = \left[\begin{bmatrix} 2 & -1 \\ 0 & 4 \\ -2 & 3 \end{bmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \; \begin{pmatrix} 2 & -1 \\ 0 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \; \begin{pmatrix} 2 & -1 \\ 0 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \; \begin{pmatrix} 2 & -1 \\ 0 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

$$= \begin{bmatrix} 0 & 7 & -2 & 1 \\ 16 & -4 & 8 & 4 \\ 8 & -9 & 6 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 4}$$

47 Dimension Compatibility

$A \in \mathbb{R}^{n \times m}$ $\qquad B \in \mathbb{R}^{s \times t}$

when is AB defined?

$$B = [b^1 \; b^2 \; \cdots \; b^t] \qquad b^i \in \mathbb{R}^s \quad i = 1, \ldots, t$$

when is Ab^i well defined?

- We need $b^i \in \mathbb{R}^m$

So AB is well defined if and only if
$m = s$.

CAUTION!! In general, $AB \neq BA$ NOT COMMUTATIVE

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 3 \end{bmatrix}$$

Not equal

Note: $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times k}$

AB well defined

BA only well defined if $k = n$

(48) The identity matrix

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$ square

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

we usually just write I with dimension by context

$$I_{n\times n} x = x \qquad \forall\, x \in \mathbb{R}^n$$

<u>Properties</u>: $A, B \in \mathbb{R}^{n\times m}$, $C, D \in \mathbb{R}^{m\times s}$, $E \in \mathbb{R}^{s\times t}$

ⓐ $A(CE) = (AC)E \in \mathbb{R}^{n\times t}$ so we just write <u>ACE</u>

ⓑ $A(C+D) = AC + AD \in \mathbb{R}^{n\times s}$

ⓒ $(A+B)C = AC + BC \in \mathbb{R}^{n\times s}$

ⓓ $\alpha(AC) = (\alpha A)C = A(\alpha C) \in \mathbb{R}^{n\times s}$ we just write αAC

ⓔ $AI_{m\times m} = I_{n\times n}A = A \in \mathbb{R}^{n\times m}$ $\quad (AI = IA = A)$

(49) Transpose of a matrix

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n , \quad X^T = (x_1, x_2, x_3, \dots, x_n)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} = \left[u^1 \ u^2 \cdots u^m \right] \in \mathbb{R}^{n \times m}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ \vdots & & & & \\ a_{1m} & a_{2m} & a_{3m} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} (u^1)^T \\ (u^2)^T \\ (u^3)^T \\ \vdots \\ (u^m)^T \end{bmatrix} \in \mathbb{R}^{m \times n}$$

ex: $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 4 & 1 & -8 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 2 & -8 \\ 3 & 1 \end{bmatrix}$

(50) Dot product and matrix vector multiplication.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = [x_1 \ x_2 \cdots x_n] \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$
$$= x^T y$$

$A \in \mathbb{R}^{n \times m}$

$$A = \begin{bmatrix} (w^1)^T \\ (w^2)^T \\ \vdots \\ (w^n)^T \end{bmatrix}, \quad Ax = \begin{bmatrix} (w^1)^T x \\ (w^2)^T x \\ \vdots \\ (w^n)^T x \end{bmatrix} = \begin{bmatrix} w^1 \cdot x \\ w^2 \cdot x \\ \vdots \\ w^n \cdot x \end{bmatrix}$$

$w^i \in \mathbb{R}^m \quad i = 1, \ldots, n$

ex: $$\begin{bmatrix} 2 & 1 & 3 & -1 \\ -2 & 0 & 5 & 4 \\ 0 & 4 & -1 & 2 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} (2 \ 1 \ 3 \ -1) \begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix} \\ (-2 \ 0 \ 5 \ 4) \begin{pmatrix} 2 \\ -1 \\ 5 \\ 2 \end{pmatrix} \\ (0 \ 4 \ -1 \ 2) \begin{pmatrix} 2 \\ -1 \\ 3 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 10 \\ 29 \\ -3 \end{pmatrix}$$

(50.5)

$$A \in \mathbb{R}^{n\times m}, \quad A^T \in \mathbb{R}^{m\times n}$$

$$A = [a^1\ a^2 \cdots a^m] = \begin{bmatrix} (v^1)^T \\ (v^2)^T \\ \vdots \\ (v^n)^T \end{bmatrix}, \quad A^T = [v^1\ v^2 \cdots v^n]$$

$$y \in \mathbb{R}^n = \mathbb{R}^{n\times 1}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad y^T = (y_1\ y_2 \cdots y_n) \in \mathbb{R}^{1\times n}$$

$$y^T \in \mathbb{R}^{1\times n},\ A \in \mathbb{R}^{n\times m} \Rightarrow y^T A \in \mathbb{R}^{1\times m}$$

$$\downarrow$$

$$y^T A = [y^T a^1\ y^T a^2 \cdots y^T a^m] \in$$

$$(y^T A)^T = \begin{bmatrix} (a^1)^T y \\ \vdots \\ (a^m)^T y \end{bmatrix} = \begin{bmatrix} a^1 \cdot y \\ \vdots \\ a^m \cdot y \end{bmatrix} = A^T y$$

order reversing!

50.6

$A \in \mathbb{R}^{n \times m}$, $B = [b^1 \cdots b^k] \in \mathbb{R}^{m \times k}$

$AB = [Ab^1 \ Ab^2 \cdots Ab^k]$

$$(AB)^T = \begin{bmatrix} (Ab^1)^T \\ (Ab^2)^T \\ \vdots \\ (Ab^k)^T \end{bmatrix} = \begin{bmatrix} (b^1)^T A^T \\ (b^2)^T A^T \\ \vdots \\ (b^k)^T A^T \end{bmatrix}$$

$$= \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_k^T \end{bmatrix} A^T$$

$$= B^T A^T$$

order reversing

51 Composition of Linear Transformations and Matrix Multiplication

$A \in \mathbb{R}^{n \times m}$, $T_A(x) = Ax \in \mathbb{R}^n$ $\quad T_A : \mathbb{R}^m \to \mathbb{R}^n$

$x \in \mathbb{R}^m$

$B \in \mathbb{R}^{m \times k}$, $T_B(y) = By \in \mathbb{R}^m$ $\quad T_B : \mathbb{R}^k \to \mathbb{R}^m$

$y \in \mathbb{R}^k$

Composition

$$\underbrace{(T_A \circ T_B)}_{\text{The composition of } T_A \text{ with } T_B}(y) = T_A(T_B(y)) \overset{!}{=} T_{AB}(y) = ABy$$

so $(T_A \circ T_B) = T_{AB}$

$(T_A \circ T_B) : \mathbb{R}^k \to \mathbb{R}^n$

$AB \in \mathbb{R}^{n \times k}$

52 ex: $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 3 & 1 \end{bmatrix}$

$$T_A(x) = Ax = \begin{bmatrix} x_1 + x_2 - 2x_3 \\ 2x_1 - x_2 + 3x_3 \end{bmatrix}, \quad T_B(y) = By = \begin{bmatrix} 2y_1 - y_2 \\ -y_1 + 4y_2 \\ 3y_1 + y_2 \end{bmatrix}$$

$$(T_A \circ T_B)(y) = T_A(T_B(y)) = T_A\begin{pmatrix} 2y_1 - y_2 \\ -y_1 + 4y_2 \\ 3y_1 + y_2 \end{pmatrix}$$

$$= \begin{bmatrix} (2y_1 - y_2) + (-y_1 + 4y_2) - 2(3y_1 + y_2) \\ 2(2y_1 - y_2) - (-y_1 + 4y_2) + 3(3y_1 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} -5y_1 & y_2 \\ 14y_1 & -3y_2 \end{bmatrix} = \underline{\begin{bmatrix} -5 & 1 \\ 14 & -3 \end{bmatrix}} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 3 & 1 \end{bmatrix} y$$

$$= T_{AB}(y)$$

(53) $A \in \mathbb{R}^{n \times m}$ $\quad B = [b^1\ b^2 \cdots b^k] \in \mathbb{R}^{m \times k}$

$$(T_A \circ T_B)(y) = T_A(T_B(y)) = T_A(By)$$

$$= A[y_1 b^1 + y_2 b^2 + \cdots + y_k b^k]$$

$$= y_1(Ab^1) + y_2(Ab^2) + \cdots + y_k(Ab^k)$$

$$= [Ab^1\ Ab^2 \cdots Ab^k]\,y$$

$$= A[b^1\ b^2 \cdots b^k]\,y$$

$$= ABy$$