§2 Chapter 3 : Matrices

Linear Transformations

We write $T : \mathbb{R}^m \to \mathbb{R}^n$ if

for every vector $x \in \mathbb{R}^m$, $T(x)$ is a vector in $\mathbb{R}^n$.

ex: $$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 + 2x_2 \\ x_2 - x_3 \\ x_1^2 + x_3^2 \end{pmatrix}, \quad T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1^2 + 2(-1) \\ (-1) - (2) \\ (1)^2 + (2)^2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

Nonlinear

Transformation = mapping

33 The transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is said to be <u>linear</u> if

$$\underline{T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)} \qquad \forall\, x, y \in \mathbb{R}^m,\ \alpha, \beta \in \mathbb{R}$$

This implies that

$$T(\alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_k x^k)$$

$$= \alpha_1 T(x^1) + \alpha_2 T(x^2) + \cdots + \alpha_k T(x^k)$$

(3.4) ex Matrix vector multiplication is an example of a linear transformation.

$$A = [a^1 \ a^2 \ \cdots \ a^m] \in \mathbb{R}^{n \times m} \quad (\text{so } a^i \in \mathbb{R}^n \ i = 1, \ldots m)$$

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

The range of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\text{Ran}(T) = \{Tx : x \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

ex: $A = [a^1 \ a^2 \cdots a^m]$, $\text{Ran}(A) = \text{Span}[a^1, a^2, \ldots, a^m]$

The nul space of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\text{Nul}(T) = \{x : Tx = 0\} \subset \mathbb{R}^m$$

ex: $\text{Nul}(A)$ = set of all solutions to the homogeneous equation $Ax = 0$

34.5

Given $A = [a^1 \cdots a^m] \in \mathbb{R}^{n \times m}$, $a^i \in \mathbb{R}^n$ $i = 1, \ldots, m$

Define $T_A : \mathbb{R}^m \to \mathbb{R}^n$ by

$$T_A(x) = Ax$$

Then T_A is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$

(35) <u>Defn</u>: Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. $\mathbb{R}$

(a) T is said to be <u>one-to-one</u> if for every $w \in \mathbb{R}^n$ there exists at most one vector $u \in \mathbb{R}^m$ such that $T(u) = w$.

(b) T is said to be <u>onto</u> if for every $w \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that $T(u) = w$.

<u>Note</u>: (i) T one-to-one <u>does not mean</u> $\exists u$ s.t. $\left(Tx \overset{?}{=} 0\right)$
$T(u) = w \quad \forall w \in \mathbb{R}^n$

(ii) T onto <u>does not mean</u> $\exists$ at most one solution to $T(u) = w \quad \forall w \in \mathbb{R}^n$.

(36) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Is T_A one-to-one? yes

Is T_A onto? No

$\nexists\, x \in \mathbb{R}^2$ s.t. $Ax = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

→ Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Is T_A one-to-one? No

Is T_A onto? yes

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Is T_A one-to-one? No

Is T_A onto? No

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$ identity matrix

(37) Let $A \in \mathbb{R}^{n \times m}$ and let $B \in \mathbb{R}^{n \times m}$ be a matrix obtained by putting A into echelon form.

Then

(i) T_A is one-to-one if and only if

(ii) T_A is onto if and only if

(3.8) When does $A = [a^1 \; a^2 \; \cdots \; a^m] \in \mathbb{R}^{n\times m}$

define a one-to-one linear transformation?

- $T_A(x) = Ax$ is one-to-one if and only if

$Ax = w$ has at most one

solution for every $w \in \mathbb{R}^n$

$\overset{\text{no free}}{\Longleftrightarrow}$ echelon form has no free variables

$\overset{\text{col piv}}{\Longleftrightarrow}$ pivot in every column

$\overset{\text{col LI}}{\Longleftrightarrow}$ columns are LI $\Longleftrightarrow$ $\text{Nul}(A) = \{0\}$

$\overset{(n,m)}{\Longrightarrow}$ $n \geq m$

(39) when does $A = [a^1\ a^2 \cdots a^m] \subset \mathbb{R}^{n\times m}$

define an onto linear tranformation?

$T_A(x) = Ax$ is onto if and only if $Ax = w$
has at least one solution for all $w \in \mathbb{R}^n$.

$\overset{\text{col space}}{\Longleftrightarrow}$ column space of $A = \mathbb{R}^n$
i.e. linear span columns of $A = \mathbb{R}^n$

$\overset{\text{Piv row}}{\Longleftrightarrow}$ echelon form must have
a pivot in every row

$\overset{(n,m)}{\Longrightarrow}$ $n \le m$

(40) When is the linear transformation T_A defined by $A = [a^1 \cdots a^m] \in \mathbb{R}^{n \times m}$ both one-to-one and onto?

(41) ex (i) $A = \begin{bmatrix} 4 & -1 \\ -2 & 2 \\ 0 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$ one-to-one
onto

one-to-one ?
onto ?

(42) $T : \mathbb{R}^m \to \mathbb{R}^n$, $u^1, u^2, \dots, u^k \in \mathbb{R}^m$

(i) If

$T(u^1), T(u^2), \dots, T(u^k) \in \mathbb{R}^n$

are linearly independent,

Are $u^1, u^2, \dots, u^k$ linearly independent? — Yes

If $\alpha_1 u^1 + \alpha_2 u^2 + \dots + \alpha_k u^k = 0$

$\Rightarrow 0 = T(0) = T(\alpha_1 u^1 + \dots + \alpha_k u^k)$

$= \alpha_1 T(u^1) + \alpha_2 T(u^2) + \dots + \alpha_k T(u^k)$

(ii) If $u^1, u^2, \dots, u^k$ are linearly independent

are $T(u^1), T(u^2), \dots, T(u^k)$

linearly independent? No

(43) Let $T : \mathbb{R}^m \to \mathbb{R}^n$.

Show that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall x, y \in \mathbb{R}^m, \ \alpha, \beta \in \mathbb{R}$$

if and only if

(i) $T(x+y) = T(x) + T(y) \quad \forall x, y \in \mathbb{R}^m$ (additivity)

and

(ii) $T(\alpha x) = \alpha T(x) \quad \forall x \in \mathbb{R}^m, \alpha \in \mathbb{R}$ (homogeneity)