(27) $u^1, \dots, u^m \in \mathbb{R}^n$, $A = [u^1, u^2, \dots, u^m] \in \mathbb{R}^{n \times m}$

$b \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, $x^1, x^2, y^1, y^2 \in \mathbb{R}^m$

(a) If $Ax^1 = b$ and $Ax^2 = b$, then

$x^1 - x^2 \in \mathrm{Nul}(A)$, i.e. $A(x^1 - x^2) = 0$

$\mathrm{Nul}(A)$
$= \{x \in \mathbb{R}^m : Ax = 0\}$
$\subset \mathbb{R}^m$

(b) If $Ay^1 = 0$ and $Ay^2 = 0$, then

$A(\alpha y^1 + \beta y^2) = 0$

Pf: (a) $A(x^1 - x^2) = Ax^1 - Ax^2 = b - b = 0$

(b) $A(\alpha y^1 + \beta y^2) = \alpha A y^1 + \beta A y^2 = \alpha \cdot 0 + \beta \cdot 0 = 0$

Every linear combination of vectors in $\mathrm{Nul}(A)$ must also be in $\mathrm{Nul}(A)$.

(29) Unifying Theorem - Version 1

$S = \{a^1, \ldots, a^n\} \subset \mathbb{R}^n \qquad A = [a^1\ a^2 \ldots a^n] \in \mathbb{R}^{n \times n}$

Then the following are equivalent.

(a) $\text{Span}[S] = \mathbb{R}^n$

(b) S is a linearly independent set of vectors

(c) $\text{Ran}[A] = \mathbb{R}^n$

(d) $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

We say $\{a^1, \ldots, a^k\} \subset \mathbb{R}^n$ span $\mathbb{R}^n$ if

$$\text{span}\{a^1, \ldots, a^k\} = \mathbb{R}^n.$$

The above theorem says that any linearly independent set of n vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$.

(30) Are the vectors

$$a^1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \quad a^2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \quad a^3 = \begin{bmatrix} -1 \\ -11 \\ 10 \end{bmatrix}$$

linearly independent?

$$\begin{array}{l|ccc}
 & 1 & 2 & -1 \\
 & -2 & 3 & -11 \\
 & 4 & -1 & 10 \\
\hline
 & 1 & 2 & -1 \\
R_2 + 2R_1 & 0 & 7 & -13 \\
R_3 - 4R_1 & 0 & -9 & 14 \\
\hline
 & 1 & 2 & -1 \\
R_3 + R_2 & 0 & -2 & 1 \\
R_2 + 3(R_3 + R_2) & 0 & 1 & -10 \\
\hline
 & 1 & 2 & -1 \\
R_2 \leftrightarrow R_2 & 0 & 1 & -10 \\
R_2 + 2R_3 & 0 & 0 & -29
\end{array}$$

Can we write a^3 as a linear combination of a^1 and a^2?

Yes

(31) (a) Is $b = \begin{pmatrix} -3 \\ 4 \\ -7 \\ 1 \end{pmatrix}$ in the linear span of

$$a^1 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \quad a^2 = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad a^3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -1 \end{pmatrix}$$

(b) are the vectors

$$a^1 = \begin{pmatrix} 9 \\ -6 \end{pmatrix}, \quad a^2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \quad a^3 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

linearly independent?

(32) Chapter 3 : Matrices

Linear Transformations

We write $T : \mathbb{R}^m \to \mathbb{R}^n$ if

for every vector $x \in \mathbb{R}^m$, $T(x)$ is a vector in $\mathbb{R}^n$.

ex: $$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 + 2x_2 \\ x_2 - x_3 \\ x_1^2 + x_3^2 \end{pmatrix}, \quad T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1^2 + 2(-1) \\ (-1) - (2) \\ (1)^2 + (2)^2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$$

Nonlinear

Transformation = mapping

(33) The transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is said to be <u>linear</u> if

$$\underline{T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)} \qquad \forall\, x, y \in \mathbb{R}^m,\ \alpha, \beta \in \mathbb{R}$$

This implies That

$$T(\alpha_1 x^1 + \alpha_2 x^2 + \cdots + \alpha_k x^k)$$

$$= \alpha_1 T(x^1) + \alpha_2 T(x^2) + \cdots + \alpha_k T(x^k)$$

(34) ex Matrix vector multiplication is an example of a linear transformation.

$$A = [a^1 \; a^2 \cdots a^m] \in \mathbb{R}^{n \times m} \quad (\text{so } a^i \in \mathbb{R}^n \;\; i = 1, \ldots m)$$

$$A(\alpha x + \beta y) = \alpha A x + \beta A y$$

The range of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\mathrm{Ran}(T) = \{ Tx : x \in \mathbb{R}^m \} \subset \mathbb{R}^n$$

ex: $A = [a^1 \; a^2 \ldots a^m]$, $\mathrm{Ran}(A) = \mathrm{Span}[a^1, a^2, \ldots, a^m]$

The nul space of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is

$$\mathrm{Nul}(T) = \{ x : Tx = 0 \} \subset \mathbb{R}^m$$

ex: $\mathrm{Nul}(A)$ = set of all solutions to the homogeneous equation $Ax = 0$

(35) Defn: Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. $\mathbb{R}$

(a) T is said to be one-to-one if for every $w \in \mathbb{R}^n$ there exists at most one vector $u \in \mathbb{R}^m$ such that $T(u) = w$.

(b) T is said to be onto if for every $w \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that $T(u) = w$.

Note: (i) T one-to-one does not mean $\exists u$ s.t. $T(u) = w \quad \forall w \in \mathbb{R}^n$ ($Tx \overset{?}{=} 0$)

(ii) T onto does not mean $\exists$ at most one solution to $T(u) = w \quad \forall w \in \mathbb{R}^n$.