Thm: $\{u^1, u^2, \dots, u^m\} \subset \mathbb{R}^n$.

If $n < m \Rightarrow \{u^1 \dots u^m\}$ are linearly Dependent

$A = [u^1\ u^2 \dots u^m] \in \mathbb{R}^{n \times m}$, $Ax = 0$, $x \neq 0$

at least $m-n$ free variable

Thm: $\{u^1, \dots, u^m\} \subset \mathbb{R}^n$ B is echelon form of $A = [u^1 \dots u^m]$ $[u^1 \dots u^m | b]$

(a) $\text{span}\{u^1 \dots u^m\} = \mathbb{R}^n$ exactly when B has a pivot position in every row.
i.e. the leading term in every row is nonzero

(b) $\{u^1, \dots, u^m\}$ are linearly independent exactly when B has a pivot position in every column

Pf (a) can solve for all rhs

(b) no free var

(2) Th: $A = [a^1\ a^2 \ldots a^m]$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

$\Rightarrow$ $\{a^1\ a^2 \ldots a^m\}$ are linearly independent if and only if the only solution to the homogeneous equation $Ax = 0$ is $x = 0$.

3

$$\begin{array}{l|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & \\
 & 2 & -6 & -1 & 8 & 7 \\
 & 1 & -3 & -1 & 6 & 6 \\
 & -1 & 3 & -1 & 2 & 4 \\
\hline
R_1 \leftrightarrow R_2 & 1 & -3 & -1 & 6 & 6 \\
R_3 + R_2 & 0 & 0 & -2 & 8 & 10 \\
R_1 - 2R_2 & 0 & 0 & 1 & -4 & -5 \\
\hline
 & 1 & -3 & -1 & 6 & 6 \\
R_3 \leftrightarrow R_2 & 0 & 0 & 1 & -4 & -5 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
R_1 + R_2 & 1 & -3 & 0 & 2 & 1 \\
 & 0 & 0 & 1 & -4 & -5 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & \uparrow & & \uparrow & \\
 & & s & & t &
\end{array}$$

Reduced Echelon Form

$\longrightarrow x_1 - 3x_2 + 0\,x_3 + 2x_4 = 1$

$x_1 = 1 + 3s - 2t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

<u>Check!</u>

$$x = \begin{pmatrix} 1 \\ 0 \\ -5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

(i) columns are <u>not</u> LI since there are free var

(ii) columns do not span since the third row has no pivot position

(4) Matrix vector multiplication

$A = \{a^1 \cdots a^m\} \in \mathbb{R}^{n \times m}$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ $\quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ $\quad , \alpha, \beta \in \mathbb{R}$

(i) $A(x+y) = Ax + Ay$

(ii) $A(x-y) = Ax - Ay$

(iii) $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$

$$A(\alpha x + \beta y) = A \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_m + \beta y_m \end{pmatrix} = (\alpha x_1 + \beta y_1) a^1 + \cdots + (\alpha x_m + \beta y_m) a^m$$

$$= \alpha x_1 a^1 + \beta y_1 a^1 + \cdots + \alpha x_m a^m + \beta y_m a^m$$

$$= \alpha [x_1 a^1 + x_2 a^2 + \cdots + x_m a^m] + \beta [y_1 a^1 + y_2 a^2 + \cdots + y_m a^m]$$

$$= \alpha Ax + \beta Ay$$

(5) Thm: $A = [a^1\ a^2 \cdots a^m] \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$ (so $a^i \in \mathbb{R}^n$ $i = 1, \ldots, m$)

Let $x^p \in \mathbb{R}^m$ be a particular solution to $Ax = b$.

Then all solutions to $Ax = b$ have the form

$$x = x^p + x^h$$

where x^h is a solution to the homogeneous equation $Ax = 0$.

Pf: Suppose x^1 and x^2 solve $Ax = b$, i.e. $Ax^1 = b$ and $Ax^2 = b$.

Then $A(x^1 - x^2) = Ax^1 - Ax^2 = b - b = 0$

Let x^p be the particular solution $Ax^p = b$

and let x be any other solution to $Ax = b$

Set $x^h = x - x^p$ so that $Ax^h = Ax - Ax^p = b - b = 0$

and $x = x^p + (x - x^p) = x^p + x^h$.

(6) Thm $\{a^1, \ldots, a^m\} \subset \mathbb{R}^n$ $A = [a^1 \ldots, a^m] \in \mathbb{R}^{n \times m}$

TFAE

(i) $\{a^1, \ldots, a^m\}$ is Linearly Independent

(ii) The vector equation

$$x_1 a^1 + x_2 a^2 + \cdots + x_m a^m = b$$

has at most one solution for every $b \in \mathbb{R}^n$

(iii) The linear system $[a^1\ a^2 \ldots a^m \mid b]$ has at most one solution for every $b \in \mathbb{R}^n$

(iv) The equation $Ax = b$ has at most one solution for every $b \in \mathbb{R}^n$.

(7) Unifying Theorem ~ Version 1

Thm: $S = \{a^1, a^2, \dots, a^n\} \subset \mathbb{R}^n$ $A = [a^1\ a^2\ \cdots\ a^n] \in \mathbb{R}^{n \times n}$

TFAE

(a) $\operatorname{span}\{S\} = \mathbb{R}^n$

(b) S is linearly independent

(c) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$