$$
\begin{aligned}
x_1 + x_2 + x_3 - 2x_4 + 4x_5 &= -5\\
-x_1 \qquad - 3x_3 + 4x_4 - 5x_5 &= 5\\
2x_1 + 4x_2 - 2x_3 + x_4 + 5x_5 &= -9
\end{aligned}
$$

$$
\begin{array}{c|ccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
 & 1 & 1 & 1 & -2 & 4 & -5\\
 & -1 & 0 & -3 & 4 & -5 & 5\\
 & 2 & 4 & -2 & 1 & 5 & -9\\
\hline
R_1 \leftrightarrow -R_2 & 1 & 0 & 3 & -4 & 5 & -5\\
R_1 + R_2 & 0 & 1 & -2 & 2 & -1 & 0\\
R_3 + 2R_2 & 0 & 4 & -8 & 9 & -5 & 1\\
\hline
 & 1 & 0 & 3 & -4 & 5 & -5\\
 & 0 & 1 & -2 & 2 & -1 & 0\\
R_3 - 4R_2 & 0 & 0 & 0 & 1 & -1 & 1\\
\hline
R_1 + 4R_3 & 1 & 0 & 3 & 0 & 1 & -1\\
R_2 - 2R_3 & 0 & 1 & -2 & 0 & 1 & -2\\
 & 0 & 0 & 0 & 1 & -1 & 1
\end{array}
$$

$$
\begin{cases}
x_1 + 3x_3 + x_5 = -1\\
x_2 - 2x_3 + x_5 = -2\\
x_4 - x_5 = 1
\end{cases}
$$

Free var

$x_3 = t$

$x_5 = s$

$$
\begin{cases}
x_1 = -1 - 3t - s\\
x_2 = -2 + 2t - s\\
x_3 = t\\
x_4 = 1 + s\\
x_5 = s
\end{cases}
$$

$$
\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{pmatrix} = \begin{pmatrix} -1\\ -2\\ 0\\ 1\\ 0 \end{pmatrix} + t\begin{pmatrix} -3\\ 2\\ 1\\ 0\\ 0 \end{pmatrix} + s\begin{pmatrix} -1\\ -1\\ 0\\ 1\\ 1 \end{pmatrix}
$$

Check by pluggin into equations

(16) 2.1 More on vectors

$\mathbb{R}^n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

$$\alpha x = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

$$x + y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

$\mathbb{R}^3$

$$x = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \quad y = \begin{pmatrix} -3 \\ -5 \\ 7 \end{pmatrix}, \quad a = 2$$

$$2x = 2\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$$

$$x + y = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ 11 \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Basic Properties of vectors

(a) $u + v = v + u$

(b) $a(u+v) = au + av$

(c) $(a+b)u = au + bu$

(d) $u - u = 0$

(e) $u + 0 = u$

(f) $1 \cdot u = u$

$\boxed{\text{Linear combination}}$

$u^1, u^2, \dots, u^k \in \mathbb{R}^n$, $c_1, c_2, \dots, c_k \in \mathbb{R}$

$c_1 u^1 + c_2 u^2 + \dots + c_k u^k$

is a linear combination of $u^1, u^2, \dots, u^k$

ex: $$2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 2 \end{pmatrix}$$

$\boxed{R\ 2.2}$

(17) Does there exist a linear combination of $u^1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $u^2 = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$, $u^3 = \begin{pmatrix} -3 \\ -5 \\ 2 \end{pmatrix}$

that equals the vector $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = b = x_1 u^1 + x_2 u^2 + x_3 u^3 = x_1 \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ -5 \\ 2 \end{pmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 - 3x_3 \\ -x_1 + 4x_2 - 5x_3 \\ 3x_1 + 2x_3 \end{bmatrix}$$

system as linear combination

Solve

$$\begin{cases} 2x_1 + x_2 - 3x_3 = 1 \\ -x_1 + 4x_2 - 5x_3 = 1 \\ 3x_1 + 2x_3 = 1 \end{cases}$$

	x_1	x_2	x_3	
$R_1 \leftrightarrow R_2$	-1	4	-5	1
	2	1	-3	1
	3	0	2	1
	1	-4	5	-1
$R_2 + 2R_1$	0	9	-13	3
$R_2 + 3R_1$	0	12	-13	4
	1	-4	5	-1
	0	9	-13	3
$R_3 - R_2$	0	3	0	1
$R_1 + R_3$	1	-1	5	0
$R_2 - 3R_3$	0	0	-13	0
$\frac{1}{3} R_3$	0	1	0	1/3

	x_1	x_2	x_3	
$R_1 + R_2$	1	0	5	1/3
R_3	0	1	0	1/3
$-\frac{1}{13} R_2$	0	0	1	0
$R_1 - 5R_3$	1	0	0	1/3
	0	1	0	1/3
	0	0	1	0

(18) <u>Linear Span</u>

Given vectors $u^1, u^2, \ldots, u^k \in \mathbb{R}^n$, the linear span of $u^1, u^2, \ldots, u^k$ is the set of all vectors that can be represented as a linear combination of $u^1, u^2, \ldots, u^k$

$$\text{Span}\{u^1, u^2, \ldots, u^k\} = \{a_1 u^1 + a_2 u^2 + \cdots + a_k u^k : a_i \in \mathbb{R} \quad i = 1, \ldots, k\}$$

$$b_1 u^1 + b_2 u^2 + \cdots + b_k u^k$$

$$(a_1+b_1) u^1 + \cdots + (a_k+b_k) u^k$$

<u>Properties</u>: (i) $0 \in \text{Span}[u^1, \ldots, u^k]$

(ii) $\alpha \in \mathbb{R},\ v \in \text{Span}[u^1, \ldots, u^k] \Rightarrow \alpha v \in \text{Span}[u^1, \ldots, u^k]$

(iii) $v^1, v^2 \in \text{Span}[u^1, \ldots, u^k] \Rightarrow v^1 + v^2 \in \text{Span}[u^1, \ldots, u^k]$

(iv) $v \in \text{Span}[u^1, \ldots, u^k] \Leftrightarrow$

$\boxed{[u^1\ u^2 \cdots u^k \mid v]}$ is consistent

<u>ex</u>: $\begin{pmatrix} 2 \\ 6 \\ -1 \\ 4 \end{pmatrix} \in \text{Span}\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -8 \\ -5 \end{pmatrix}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 2 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & -8 & -1 \\ 1 & 1 & -5 & 4 \end{array}\right]$ is consistent

(False)

(19) Given $u^1, u^2, \dots, u^k \in \mathbb{R}^n$ we can form an associated $n \times k$ matrix

$$A = \left[u^1 \; u^2 \; \dots \; u^k \right] \in \mathbb{R}^{n \times k}$$

ex: $u^1 = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix}$ $u^2 = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$. $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 7 & 8 \end{bmatrix}$ —— $A \in \mathbb{R}^{n \times k}$, $n = 3, k = 2$

Fact: Suppose B is the $n \times k$ matrix obtained by transforming A to echelon form.
If the leading coefficient in every row of B is nonzero, then $\mathrm{Span}[u^1, u^2, \dots, u^k] = \mathbb{R}^n$

Defn: $A = [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$

$$\left.\begin{array}{l} a_{i_0 j_0} \text{ is a pivot position for } A \\ \text{if } a_{i_0 j_0} \neq 0 \quad a_{ij} = 0 \quad \begin{array}{l} i < i_0 \\ j \geq j_0 \end{array} \end{array}\right\} \text{echelon form}$$

20 Matrix - vector Multiplication

$\mathbb{R}^{n\times m} \ni A = [a^1 \; a^2 \; \dots \; a^m] \qquad a^i \in \mathbb{R}^n \quad i = 1 \dots n$

$x \in \mathbb{R}^m, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

$$\boxed{\frac{Ax = b}{A \in \mathbb{R}^{s\times t}} \quad \begin{matrix} b \in \mathbb{R}^s \\ x \in \mathbb{R}^t \end{matrix}}$$

$$Ax = x_1 a^1 + x_2 a^2 + x_3 a^3 + \dots + x_n a^m$$

ex: $A = \begin{bmatrix} 2 & 1 & 7 & 0 \\ -1 & -2 & -1 & 8 \\ 4 & 3 & -5 & -1 \end{bmatrix}, \quad x = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 4 \end{pmatrix}$

$$Ax = -\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 7 \\ -1 \\ -5 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 35 \\ -21 \end{pmatrix}$$

$$\begin{aligned} 2x_1 + x_2 + 7x_3 &= 10 \\ -x_1 - 2x_2 - x_3 + 8x_4 &= 35 \\ 4x_1 + 3x_2 - 5x_3 - x_4 &= -21 \end{aligned}$$

$x = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 4 \end{pmatrix}$ solves

are there other solutions?

(21) **Theorem**: square □ short + fat = $n \times m$, $n < m$ tall skinny $n > m$

$a^1, a^2, \ldots, a^m \in \mathbb{R}^n$ $b \in \mathbb{R}^n$, $A = [a^1\ a^2 \cdots a^m] \in \mathbb{R}^{n \times m}$

The following statements are equivalent, i.e., if any one of them is true, then all of them are true.

TFAE

(a) $b \in \text{Span}\{a^1, a^2, \ldots, a^m\}$

(b) the vector equation

$$x_1 a^1 + x_2 a^2 + \cdots + x_m a^m = b$$

has at least one solution.

(c) The linear system $[a^1\ a^2\ \cdots\ a^m \mid b] \to [A \mid b]$ has at least one solution

(d) the matrix equation $Ax = b$ has at least one solution.

Defn: The range of A = $\text{Span}\{a^1\ a^2\ \ a^m\}$ = $\text{Ran}(A)$.

~~(also Im(A))~~

(22) $\boxed{2.3}$ Linear Independence $\{u^1\ u^2 \ldots u^m \mid 0\}$

<u>Defn</u>: Let $\{u^1, u^2, \ldots, u^m\} \subset \mathbb{R}^n$.

If the only solution to the vector equation

$$x_1 u^1 + x_2 u^2 + \cdots + x_m u^m = 0 \qquad \text{(Homogeneous System)}$$

is given by $x_1 = x_2 = x_3 = \cdots = x_m = 0$, then we say that the set of vectors $\{u^1, u^2, \ldots, u^m\}$ are linearly independent.

Checking Linear Independence: solve $Ax = 0$

$$\left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ -2 & 7 & 9 & 0 \\ -3 & 7 & -5 & 0 \\ 4 & -13 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} -R_1 \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4+4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 13 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & -7 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3+2R_2 \\ R_4+R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 13 & 0 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \frac{1}{6}R_4 \\ R_3-4R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 13 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yes - linear independent

(23) Thm: $\{u^1, u^2, \ldots, u^m\} \subset \mathbb{R}^n$.

If $n < m$, then the set is linearly dependent

Pf: $A = [u^1 \ u^2 \ \cdots \ u^m] = \boxed{\qquad\qquad}_{n \times m}$ $\quad n < m$

Echelon Form can have at most n leading terms (pivot positions) so there are at least $m - n > 0$ free variables. Each free variable is associated with a nontrivial solution to the homogeneous equation $Ax = 0$. Hence the vectors are linearly dependent.

- If there is a nonzero vector x solving $Ax = 0$, then the columns of A are linearly dependent.

(24) Thm: $\{u^1, \dots, u^m\} \subset \mathbb{R}^n$ are linearly dependent if and only if one of the vectors in this set is in the linear span of the remaining vectors.

Pf: Lin Dep $\Rightarrow \exists\, a_1, \dots, a_m$ with $a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \neq 0$ s.t.

$$a_1 u^1 + \dots + a_m u^m = 0$$

$$a \neq 0 \Rightarrow \exists\, i_0 \text{ s.t. } a_{i_0} \neq 0 \Rightarrow u^{i_0} = -\frac{a_1}{a_{i_0}} u^1 - \dots - \frac{a_{i_0-1}}{a_{i_0}} u^{i_0-1} - \frac{a_{i_0+1}}{a_{i_0}} u^{i_0+1} - \dots - \frac{a_m}{a_{i_0}} u^m$$

$$u^{i_0} = \sum_{i \neq i_0} b_i u^i \Rightarrow 0 = -u^{i_0} + \sum_{i \neq i_0} b_i u^i$$

(25) Give a lin. dep set of vectors

$$u^1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u^2 = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}, \quad u^3 = u^1 + u^2 = \begin{pmatrix} 5 \\ 0 \\ 8 \end{pmatrix}$$

Th: $u^1, \dots, u^m \in \mathbb{R}^n$, $A = [u^1 \ u^2 \ u^m] \in \mathbb{R}^{n \times m}$

$B \in \mathbb{R}^{n \times m}$ is A transformed to echelon form

Then

(a) $\mathrm{Span}\{u^1, \dots, u^k\} = \mathbb{R}^n$ exactly when B has a pivot position in every row, i.e. every row has a nonzero leading term.

(b) $\{u^1, \dots, u^k\}$ are linearly independent exactly when B has a pivot position in every column.

pf. (a) $\Leftrightarrow$ we can solve for every possible right hand side

(b) $\Leftrightarrow$ there are no free variables so the only solution to $Ax = 0$ is $x = 0$.

(26) Homogeneous linear systems

$u^1, \dots, u^m \in \mathbb{R}^m \quad A = [u^1 \dots u^m]$

$Ax = 0$ homogeneous linear system

$Nul(A) = \{x : Ax = 0\}$ ~ the null space of A

(also $\ker(A)$ = the kernel of A = null space of A)

Thm: $\{u^1, \dots, u^m\}$ are linearly independent if and only if $Nul(A) = \{0\}$.

Matrix Vector Multiplication

$A = [a^1\ a^2 \dots a^m] \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \qquad \alpha \in \mathbb{R}, \beta \in \mathbb{R}$

(a) $A(x+y) = Ax + Ay$

(b) $A(x-y) = Ax - Ay$

(c) $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$