1.2

(10) Solving linear Systems

- Two linear systems are said to be equivalent if they have exactly the same solution set
- Grand strategy: Manipulate the given linear system so as to obtain an equivalent linear system in echelon form. Then parameterize the free variable and back solve to obtain a description of the solution set.

Manipulation Rules: ⇒ 3 elementary operations

(1) Interchange the position of any two equations

(2) multiply any equation by a nonzero constant

(3) add an equation to a multiple of another (*)

The process of applying the 3 elementary operations to a system to obtain an equivalent system in echelon form is called Gaussian Elimination

Geometric Interpretation.

(11) ex

$$\begin{aligned} -4x_1 + 2x_2 - 2x_3 &= 10 \\ x_1 + x_3 &= -3 \\ 3x_1 - x_2 + x_3 &= -8 \end{aligned}$$

$R_1 \leftrightarrow R_2$
$$\left\{\begin{aligned} x_1 + x_3 &= -3 \\ -4x_1 + 2x_2 - 2x_3 &= 10 \\ 3x_1 - x_2 + x_3 &= -8 \end{aligned}\right.$$

$$\begin{array}{llr} R_1 & x_1 + x_3 & = -3 \\ R_2 + 4R_1 & 2x_2 + 2x_3 & = -2 \\ R_3 - 3R_1 & -x_2 - 2x_3 & = 1 \end{array}$$

$$\begin{array}{llr} R_1 & x_1 + x_3 & = -3 \\ \frac{1}{2}R_2 & x_2 + x_3 & = -1 \\ R_2 + 2R_3 & -2x_3 & = 0 \end{array}$$

$x_1 = -3 - x_3 = -3$

$x_2 = -1 - x_3 = -1$.

$x_3 = 0$

$, x = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

$$\begin{array}{llr} R_1 + \frac{1}{2}R_3 & x_1 & = -3 \\ R_2 + \frac{1}{2}R_3 & x_2 & = -1 \\ -\frac{1}{2}R_3 & x_3 & = 0 \end{array}$$

Gauss-Jordan Elimination

Check: $-4(-3) + 2(-1) - 2(0) \overset{\checkmark}{=} 10$

$-3 + 0 \overset{\checkmark}{=} -3$

$3(-3) - (-1) + 0 \overset{\checkmark}{=} -8$

(12) Augmented Matrices : A book keeping technique

$$-4x_1 + 2x_2 - 2x_3 = 10$$
$$x_1 + x_3 = -3$$
$$3x_1 - x_2 + x_3 = -8$$

Augmented Matrix

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}\left[\begin{array}{ccc|c} -4 & 2 & -2 & 10 \\ 1 & 0 & 1 & -3 \\ 3 & -1 & 1 & -8 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$x_1 + x_3 = -3$$
$$-4x_1 + 2x_2 - 2x_3 = 10$$
$$3x_1 - x_2 + x_3 = -8$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ -4 & 2 & -2 & 10 \\ 3 & -1 & 1 & -8 \end{array}\right]$$

$R_2 + 4R_1$

$R_3 - 3R_1$

$$x_1 + x_3 = -3$$
$$2x_2 + 2x_3 = -2$$
$$-x_2 - 2x_3 = 1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$\frac{1}{2}R_2$

$R_3 + \frac{1}{2}R_2$

$$x_1 + x_3 = -3$$
$$x_2 + x_3 = -1$$
$$-x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Echelon Form

$-R_3$

$$x_1 + x_3 = -3$$
$$x_2 + x_3 = -1$$
$$x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Elementary row operations
1. interchange any two rows
2. multiply a row by a nonzero constant
3. replace a row by the sum of the row and a multiple of another row

$x_3 = 0$

$x_2 = -1 - x_3 = -1$

$x_1 = -3 - x_3 = -3$

(13)

$$-3x_1 + 2x_2 - x_3 + 6x_4 = -7$$
$$7x_1 - 3x_2 + 2x_3 - 11x_4 = 14$$
$$x_1 \qquad\qquad - x_4 = 1$$

Gaussian Elimination

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ -3 & 2 & -1 & 6 & -7 \\ 7 & -3 & 2 & -11 & 14 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 + 3R_1 \\ R_3 - 7R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 2 & -1 & 3 & -4 \\ 0 & -3 & 2 & -4 & 7 \end{array}\right]$$

$$-R_2 - R_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & -3 & 2 & -4 & 7 \end{array}\right]$$

$$R_3 + 3R_2 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & -1 & -1 & -2 \end{array}\right]$$

$$-R_3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

Back substitute

$$x_4 = t$$
$$x_3 = 2 - x_4 = 2 - t$$
$$x_2 = -3 + x_3 - x_4$$
$$= -3 + (2 - t) - t$$
$$= -1 - 2t$$
$$x_1 = 1 + x_4 = 1 + t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1 + t \\ -1 - 2t \\ 2 - t \\ t \end{bmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix}$$

Check: Plug in $\begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$

$$-3(1) + 2(-1) - 2 \overset{\checkmark}{=} -7$$
$$7(1) - 3(-1) + 2(2) \overset{\checkmark}{=} 14$$
$$(1) \overset{\checkmark}{=} 1$$

Plug in $\begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix}$

$$-3(1) + 2(-2) - (-1) + 6(1) \overset{\checkmark}{=} 0$$
$$7(1) - 3(-2) + 2(-1) - 11(1) \overset{\checkmark}{=} 0$$
$$(1) \qquad - (1) \overset{\checkmark}{=} 0$$

(14) Gauss-Jordan Elimination

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_4 = t \\ x_1 - x_4 = 1 \\ x_2 + 2x_4 = -1 \\ x_3 + x_4 = 2 \end{array} \Rightarrow \begin{array}{l} x_1 = 1 + x_4 = 1 + t \\ x_2 = -1 - 2x_4 = -1 - 2t \\ x_3 = 2 - x_4 = 2 - t \end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ -1 \\ 1 \end{pmatrix} \Leftarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad x_4 = t$$

Reduced Echelon Form

A matrix is in reduced echelon form if

(a) it is in echelon form,

(b) the leading coefficient in each row is 1

(c) the only nonzero entry in a pivot column is the pivot.

The pivot position in a row is the position of the leading term in each row.

A pivot column is the column associated with a pivot.

(15)
$$\begin{aligned} x_1 + x_2 + x_3 - 2x_4 + 4x_5 &= -5 \\ -x_1 \quad - 3x_3 + 4x_4 - 5x_5 &= 5 \\ 2x_1 + 4x_2 - 2x_3 + x_4 + 5x_5 &= -9 \end{aligned}$$

$$\begin{array}{c} \\ \\ \\ \\ R_1 \leftrightarrow -R_2 \\ R_1 + R_2 \\ R_3 + 2R_2 \\ \\ \\ R_3 - 4R_2 \\ R_1 + 4R_3 \\ R_2 - 2R_3 \\ \\ \end{array}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & 1 & 1 & -2 & 4 & -5 \\ -1 & 0 & -3 & 4 & -5 & 5 \\ 2 & 4 & -2 & 1 & 5 & -9 \\ \hline 1 & 0 & 3 & -4 & 5 & -5 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 4 & -8 & 9 & -5 & 1 \\ \hline 1 & 0 & 3 & -4 & 5 & -5 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ \hline 1 & 0 & 3 & 0 & 1 & -1 \\ 0 & 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ \hline \end{array}$$

Free var.
$x_3 = t$
$x_5 = s$

$$\begin{cases} x_1 + 3x_3 + x_5 = -1 \\ x_2 - 2x_3 + x_5 = -2 \\ x_4 - x_5 = 1 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t - s \\ x_2 = -2 + 2t - s \\ x_3 = t \\ x_4 = 1 + s \\ x_5 = s \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Check by pluggin into equations

(15.5) Check

$$\checkmark \left\{\begin{array}{l} (-1) + 1\cdot(-2) + (0) - 4(1) + 4(0) = -5 \\ -(-1) + 0(-2) - 3(0) + 4(1) - 5(0) = 5 \\ 2(-1) + 4(-2) - 2(0) + (1) + 5(0) = -9 \end{array}\right.$$

$$\checkmark \left\{\begin{array}{l} (-3) + (2) + (1) = 0 \\ -(-3) \quad - 3(1) = 0 \\ 2(-3) + 4(2) - 2(1) = 0 \end{array}\right.$$

$$\checkmark \left\{\begin{array}{l} (-1) + (-1) \quad - 2(1) + 4(1) = 0 \\ -(-1) + 0(-1) \quad + 4(1) - 5(1) = 0 \\ 2(-1) + 4(-1) \quad + (1) \quad 5(1) = 0 \end{array}\right.$$