(6) triangular Systems (upper triangular)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots &+ a_{1n}x_n = b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots &+ a_{2n}x_n = b_2 \\
+ a_{33}x_3 + \cdots &+ a_{3n}x_n = b_3 \\
&\vdots \\
a_{nn}x_n &= b_n
\end{aligned}
$$

n eq
n vars

$a_{ii} \neq 0$
$i = 1, \ldots, n$

$a_{ij} = 0 \quad i < j$

$a_{jj} \neq 0 \quad j = 1, \ldots, n$

ex:

$$
\begin{aligned}
2x_1 \quad + 2x_3 \quad &= 12 \\
12x_2 \quad - 5x_4 &= 19 \\
3x_3 + 11x_4 &= 15 \\
- x_4 &= 3
\end{aligned}
$$

Properties

(1) every variable is leading in exactly one eq
(2) same # eq as vars
(3) exactly one solution

solution technique: Back substitution

$x_4 = -3$

$x_3 = \frac{1}{3}[15 - 11x_4] = \frac{1}{3}[15 + 11 \cdot 3] = \frac{1}{3}[15 + 33] = \frac{48}{3} = 16$

$x_2 = \frac{1}{12}[19 + 5x_4] = \frac{1}{12}[19 - 5 \cdot 3] = \frac{1}{12} \cdot 4 = \frac{1}{3}$

$x_1 = \frac{1}{2}[12 - 2x_3] = \frac{1}{2}[12 - 2 \cdot 16] = \frac{1}{2}[12 - 32] = -10$

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = X = \begin{pmatrix} -10 \\ 1/3 \\ 16 \\ -3 \end{pmatrix}
$$

check

(7) Echelon Systems

A linear system is in echelon form (called an echelon system) if

(i) it is organized in a descending staircase pattern from left to right

(ii) the index of the leading variable in each equation is strictly increasing from top to bottom

(iii) the equations of the form $0 = c$ are at the bottom with those where $c \neq 0$ above those where $c = 0$

ex:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 1 \\ x_2 + 2x_3 - 2x_4 &= 2 \\ x_4 &= 1 \end{aligned}$$

3 equations in 4 variables

staircase

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_2 + 2x_3 &= 4 \end{aligned}$$

→ x_3 is called a free variable since it is never a leading variable.

Back substitution (Back solve)

$x_3 = t \Rightarrow x_2 = 4 - 2t$

$x_1 = x_3 - 2x_2 = t - 2(4-2t) = -8 + 5t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -8+5t \\ 4-2t \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

a line in $\mathbb{R}^4$

check

(8) Properties of Echelon Systems

(i) every variable leads in at most one equation

(ii) the solution set is (a) empty, (b) exactly one solution, or (c) has infinitely many solutions

- triangular systems are in echelon form and have exactly one solution.

- Any variable in a system in echelon form that is not leading in any equation is called a free variable.

In solving echelon systems we assign a parameter to each free variable and then use back substitution to obtain a description of the solution set

(9) ex.

$$\begin{aligned} -7x_1 + 3x_2 \quad + 8x_4 - 2x_5 + 13x_6 &= -6 \\ -5x_3 - x_4 + 6x_5 + 3x_6 &= 0 \\ 2x_4 + 5x_5 &= 1 \end{aligned}$$

free variable : x_2, x_5, x_6 , $x_2 = r$, $x_5 = s$, $x_6 = t$

Back solve

$$x_4 = \tfrac{1}{2}(1 - 5x_5) = \tfrac{1}{2} - \tfrac{5}{2}s$$

$$x_3 = -\tfrac{1}{5}[x_4 - 6x_5 - 3x_6] = -\tfrac{1}{5}(\tfrac{1}{2} - \tfrac{5}{2}s - 6s - 3t]$$

$$= -\tfrac{1}{10} + \tfrac{17}{10}s + \tfrac{3}{5}t$$

$$x_1 = -\tfrac{1}{7}[-6 - 3x_2 - 8x_4 + 2x_5 - 13x_6] = -\tfrac{1}{7}[-3r - 8(\tfrac{1}{2} - \tfrac{5}{2}s) + 2s - 13t]$$

$$= \tfrac{6}{7} + \tfrac{3}{7}r + \tfrac{4}{7}(1 - 5s) - \tfrac{2}{7}s + \tfrac{13}{7}t = \tfrac{6}{7} + \tfrac{3}{7}r + \tfrac{4}{7} - \tfrac{22}{7}s + \tfrac{13}{7}t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = X = \begin{bmatrix} \frac{10}{7} + \frac{3}{7}r - \frac{22}{7}s + \frac{13}{7}t \\ r \\ -\frac{1}{10} + \frac{17}{10}s + \frac{3}{5}t \\ \frac{1}{2} - \frac{5}{2}s \\ s \\ t \end{bmatrix} = \begin{pmatrix} \frac{10}{7} \\ 0 \\ -1/10 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} \frac{3}{7} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -22/7 \\ 0 \\ 17/10 \\ -5/2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{13}{7} \\ 0 \\ \frac{3}{5} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Check