① Ex: A 60 gallon bathtub is to be filled with water that is exactly 100°F.
The hot water supply is 120°F and the cold is 60°F.
Assume that when mixed, the temperature of the water will be the weighted averages by gallons from each water source.
How much hot + cold water should be used to achieve the 100°F goal?

h - gallons of hot $\quad$ c - gallons of cold

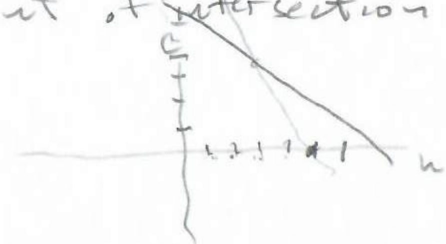
$$h + c = 60$$
$$120\frac{h}{60} + 60\frac{c}{60} = 100$$

$$h + c = 60$$
$$2h + c = 100$$
$$h = 40 \Rightarrow c = 20$$

what if hot is 125°F?

$$h + c = 60$$
$$\frac{25}{12}h + c = 100$$
$$\frac{13}{12}h = 40 \Rightarrow h = \frac{480}{13} \qquad c = 60 - \frac{480}{13} = \frac{780}{13} - \frac{480}{13} = \frac{300}{13}$$

Geometrically $(h, c) = (40, 20)$ is the point of intersection of the two lines $h + c = 60$, $2h + c = 100$



(2) Quick review of vectors

$\mathbb{R}$ ~ real numbers

$\mathbb{R}^2$ ~ $\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \}$ $\qquad$ $\mathbb{R}^n = \{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{R}, i = 1..n \}$

$\mathbb{R}^3$ ~ $\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R} \}$

$x, y \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ $\qquad$ x_i's components of x

y_i's components of y

i ~ index

$x = y \iff x_i = y_i \quad i = 1 \ldots n$

x_{i_0} has index i_0

$x \neq y \iff \exists \; i_0 \in \{1, \ldots, n\}$ s.t. $x_{i_0} \neq y_{i_0}$

magnitude:

$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $\qquad$ magnitude = Euclidean norm

unit vectors : $u \in \mathbb{R}^n$ s.t. $\|u\| = 1$

unit coordinate vectors : $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ coordinate

Dot Product

$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i = x^T y$ $\quad$ later

$= \underline{\|x\| \|y\| \cos\theta}$

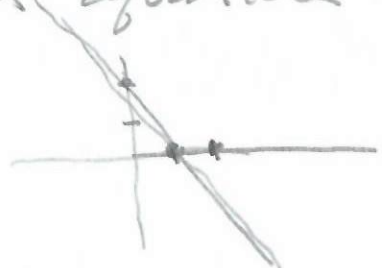
where θ is the smallest angle between x & y.

(3) Linear Equations

$\mathbb{R}^2$, $a_1 x_1 + a_2 x_2 = b$, $a_1, a_2, b \in \mathbb{R}$

ex: $2x_1 + x_2 = 2$

geometrically, the set of all solutions of this equation is a line in $\mathbb{R}^2$



if $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq 0$

If $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$ and $b \neq 0$, then the solution set is the empty set $\emptyset$

in this case we say that the equation $a_1 x_1 + a_2 x_2 = b$ is inconsistent.

$\mathbb{R}^1$: $a_1 x_1 = b$, $a_1, b \in \mathbb{R}$ if $a_1 \neq 0$. set of solutions is a point

$\mathbb{R}^3$: $a_1 x_1 + a_2 x_2 + a_3 x_3 = b$, $a_1, a_2, a_3, b \in \mathbb{R}$ if $a \neq 0$ set of solutions is a plane

$\mathbb{R}^n$: $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$, $a_i \in \mathbb{R}$ $\forall i$ if $a \neq 0$ set of solutions is called a hyperplane

hyperplane divides $\mathbb{R}^n$ in half

(4) Systems of Linear Equations

ex: $\boxed{\begin{aligned} h + c &= 60 \\ 2h + c &= 100 \end{aligned}}$ linear system of 2 eq in 2 variables

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + \quad\; + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

linear system of m equations in n variables

system is <u>consistent</u> if $\exists$ at least one solution

<u>Inconsistent</u> if solution set is empty

ex:
$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ x_1 - x_2 + x_3 &= 1 \end{aligned}$$
$m = 2 \quad n = 3$ ij indices

Describe the set of solutions: geometrically, the intersection of 2 planes in $\mathbb{R}^3$, a line or the planes are parallel + don't intersect, $\emptyset$

add

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ 3x_1 \qquad\quad &= 5 \end{aligned}$$

$x_1 = 5/3 \Rightarrow x_2 - x_3 = 4 - \frac{10}{3} = \frac{2}{3}$

$x_3 = t$

$x_2 = \frac{2}{3} + t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 2/3 + t \\ t \end{pmatrix} = \begin{pmatrix} 5/3 \\ 2/3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(5) $6x_1 - 5x_2 = 4$
$9x_1 + k x_2 = -1$

For what values of k is this a consistent system?

$3x_1 + (5+k)x_2 = -5$
$6x_1 - 5x_2 = 4$

$3x_1 + (5+k)x_2 = -5$
$0 \quad -(2k+15)x_2 = 14$

$\Rightarrow$ need $2k+15 \neq 0$
or $k \neq -15/2$

solution is $x_2 = \frac{-14}{2k+15}$

$x_1 = \frac{1}{3}\left[-(5+k)x_2 - 5\right] = \frac{1}{3}\left[14\frac{5+k}{2k+15} - 5\right]$

ex: $k=0 \Rightarrow x_1 = \frac{1}{3}\left[\frac{14}{3} - 5\right] = \frac{1}{3}\cdot\frac{-1}{3} = -\frac{1}{9}$

$x_2 = \frac{-14}{15}$

Observe for $k = -15/2$ $\quad 6x_1 - 5x_2 = 4$
$9x_1 - \frac{15}{2}x_2 = -1$

parallel lines
$\begin{pmatrix}6\\-5\end{pmatrix} = \frac{2}{3}\begin{pmatrix}9\\-15/2\end{pmatrix} = \begin{pmatrix}6\\-5\end{pmatrix}$

when is the system
$ax + by = c$
$dx + fy = h$
inconsistent

(i) $\begin{pmatrix}a\\b\end{pmatrix} = 0 \quad c \neq 0$
(ii) $\begin{pmatrix}d\\f\end{pmatrix} = 0 \quad h \neq 0$
(iii) parallell and don't intersect
ie $\begin{pmatrix}a\\b\end{pmatrix} = t\begin{pmatrix}d\\f\end{pmatrix}$ but $c \neq th$