(1) Evaluate the following indefinite integrals.

(a) (20 points) \( \int \frac{\cos \sqrt{t}}{\sqrt{t}} \, dt \).

**Solution:** Use the substitution \( u = \sqrt{t} \) to get \( du = \frac{dt}{2\sqrt{t}} \). We then have

\[
\int \frac{\cos \sqrt{t}}{\sqrt{t}} \, dt = 2 \int \cos u \, du = 2 \sin u + C = 2 \sin(\sqrt{t}) + C .
\]

(b) (20 points) \( \int \frac{\cos x}{\sin^2 x + 1} \, dx \).

**Solution:** Use the substitution \( u = \sin x \) to get \( du = \cos x \, dx \). We then have

\[
\int \frac{\cos x}{\sin^2 x + 1} \, dx = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C = \tan^{-1}(\sin x) + C .
\]

(c) (10 points) \( \int \frac{x^2}{\sqrt{x+1}} \, dx \).

**Solution:** Use the substitution \( u = x + 1 \) to get \( du = dx \). Since \( x = u - 1 \), we have \( x^2 = (u - 1)^2 = u^2 - 2u + 1 \). and so

\[
\int \frac{x^2}{\sqrt{x+1}} \, dx = \int \frac{u^2 - 2u + 1}{\sqrt{u}} \, du = \int u^{3/2} - 2u^{1/2} + u^{-1/2} \, du = \frac{u^{5/2}}{5/2} - 2 \frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + C = \frac{2}{5} u^{5/2} - \frac{4}{3} u^{3/2} + 2u^{1/2} + C
\]

\[
= \frac{2}{5} (x + 1)^{5/2} - \frac{4}{3} (x + 1)^{3/2} + 2(x + 1)^{1/2} + C
\]

(2) Evaluate the following definite integrals.

(a) (20 points) \( \int_0^1 (3 + x\sqrt{x}) \, dx \)

**Solution:**

\[
\int_0^1 (3 + x\sqrt{x}) \, dx = 3 \int_0^1 dx + \int_0^1 x^{3/2} \, dx = 3 + \left[ \frac{x^{5/2}}{5/2} \right]_0^1 = 3 + \frac{2}{5} = \frac{17}{5} .
\]

(b) (20 points) \( \int_1^{e^2} \frac{(\ln x)^2}{x} \, dx \)

**Solution:** Use the substitution \( u = \ln x \) to get \( du = \frac{dx}{x} \). Since \( u = 0 \) when \( x = 1 \) and \( u = 2 \) when \( x = e^2 \), we have

\[
\int_1^{e^2} \frac{(\ln x)^2}{x} \, dx = \int_0^2 u^2 \, du = \left[ \frac{u^3}{3} \right]_0^2 = \frac{8}{3} .
\]
(c) (10 points) \[ \int_0^1 x^3 \sqrt{1 - x^2} \, dx \]

Solution: Use the substitution \( u = 1 - x^2 \) to get \( du = -2x \, dx \). With this substitution, \( x^3 \, dx = \frac{-1}{2} (x^2) (-2x \, dx) = \frac{-1}{2} (1 - u) \, du \), and

\[ u = 0 \text{ when } x = 1 \quad \text{and} \quad u = 1 \text{ when } x = 0. \]

Hence,

\[
\int_0^1 x^3 \sqrt{1 - x^2} \, dx = -\frac{1}{2} \int_1^0 (1 - u) \sqrt{u} \, du = \frac{1}{2} \int_0^1 u^{1/2} - u^{3/2} \, du
\]

\[ = \frac{1}{2} \left[ \frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2} \right]_0^1 = \frac{1}{2} \left[ \frac{2}{3} - \frac{2}{5} \right] = \frac{2}{15}. \]

(3) Differentiate the following functions.

(a) (25 points) \( F(x) = \int_1^{2x} \frac{t^2 - 1}{t^4 + 1} \, dt \)

Solution: \( F'(x) = 2 \frac{(2x)^2 - 1}{(2x)^4 + 1}. \)

(b) (25 points) \( G(x) = \int_{\cos x}^{e^{x^2}} \sec u \, du \)

Solution: \( G'(x) = 2xe^{x^2} \sec(e^{x^2}) + \sin x \sec(cos x). \)

(4) Water flows into and out of a storage tank. A graph of the rate of change \( r(t) \) of the volume of the water in the tank, in liters per day, is shown. Assume the amount of water in the tank at time \( t = 0 \) is 5,000 L.

\[
\begin{array}{c|cccc|c}
\text{Time (t)} & 0 & 0.5 & 1 & 1.5 & 2 & 2.5 & 3 & 3.5 & 4 \\
\hline
\text{Rate (r)} & -1000 & -500 & 0 & 500 & 1000 & 1500 & 2000 & \end{array}
\]

(a) (15 points) Use a Riemann Sum with \( \Delta t = 1 \) to give an upper bound for the amount of water in the tank 4 days later.

Solution: Upper Bound = 20000 + 1800 + 2000 + 1800 + 500 = 26,100 liters.
(b) (15 points) Use a Riemann Sum with $Δt = 1$ to give a lower bound for the amount of water in the take 4 days later.

**Solution:** Lower Bound $= 20000 + 500 + 1750 + 500 - 800 = 21950$.

(c) (20 points) Use the Midpoint Rule to estimate the amount of water in the take 4 days later.

**Solution:** Midpoint Rule Estimate $= 20000 + 1250 + 2000 + 1250 - 250 = 24250$.

(5) In this problem we consider the motion of a falling body with gravity being the only force acting on the body. Recall that the acceleration due to gravity is $−32 \text{ ft/sec}^2$.

(a) (20 points) If the initial velocity at time $t = 0$ is $v_0$ and the initial height above ground at time $t = 0$ is $h_0$, give a formula for $h(t)$ the height above ground at time $t$?

**Solution:**

Acceleration: $a(t) = -32 \text{ ft/s}^2$.

Velocity: $\int a(t) \, dt = -32t + C$, $v(0) = v_0 \Rightarrow C = v_0$.

Height: $h(t) = \int v(t) \, dt = \int -32t + v_0 \, dt = -16t^2 + v_0t + D$.

We have

$v(t) = -32t + v_0 \quad \text{and} \quad h(t) = -16t^2 + v_0t + h_0$.

(b) (15 points) Assuming $v_0 \geq 0$, at what time does the falling body attain its maximum height above ground?

**Solution:** Since $h(t)$ is concave down ($h''(t) = -32$), $h(t)$ attains its maximum value when $0 = h'(t) = v(t) = -32t + v_0$. That is, the time $t_m$ at which the maximum height is attained satisfies $0 = -32t_m + v_0$, or equivalently,

$$t_m = \frac{v_0}{32} \geq 0.$$ 

(c) (15 points) A ball is thrown up in the air from an initial height of $h_0$ feet. After 3 seconds it reaches its maximum height, and after 10 seconds it hits the ground. What is $h_0$?

**Solution:** By Part (b) above, we know that $3 = t_m = v_0/32$ so that $v_0 = 96$. By Part (a) we now know that $h(t) = -16t^2 + 96t + h_0$. Furthermore, at time $t = 10$ we have $h(10) = 0$, and so

$0 = h(10) = -16(10)^2 + 96(10) + h_0 = -1600 + 960 + h_0 = -640 + h_0$.

Therefore, $h_0 = 640 \text{ ft}$. 