MY FAVORITE OPEN PROBLEMS

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This note offers a collection of open problems, mostly about Brownian motion. I tried to solve every problem on the list but I failed to do so. Although I do not recall seeing these problems published anywhere, I do not insist on being their author—other mathematicians might have proposed them independently.

1. Probabilistic version of McMillan's Theorem in higher dimensions. Let X_t be the d-dimensional Brownian motion starting from the origin, and let D be an open set in the d-dimensional space containing the origin. Let $\tau = \inf\{t > 0 : X_t \notin D\}$ be the exit time from D. Consider the set A of "asymptotic directions of approach," depending on the domain D and the trajectory of X_t, and defined as the set of all cluster points of

$$\frac{X_t - X_\tau}{|X_t - X_\tau|},$$

as $t \uparrow \tau$. It has been proved in Burdzy (1990a) that for $d = 2$, a.s., the set A is equal to either a circle or a semicircle. In some domains D, the set A is a circle with a non-trivial probability, i.e., with a probability strictly between 0 and 1.

Problem 1. *Is it true that for every $d > 2$ and every d-dimensional open set D, the set A of asymptotic directions of approach is either a sphere or a hemisphere, a.s.?*

2. Topology of planar Brownian trace. Let X_t be the two-dimensional Brownian motion. A Jordan arc is a set homeomorphic to a line segment.

Problem 2 (i). *Is it true that for every pair of points $x, y \notin X[0,1]$ one can find a Jordan arc Γ with $x, y \in \Gamma$, and such that $\Gamma \cap X[0,1]$ contains only a finite number of points, possibly depending on x and y?*

Let $\{A_k\}_{k\geq 1}$ be the family of all connected components of the complement of $X[0,1]$, and let $K = X[0,1] \setminus \bigcup_{k\geq 1} \partial A_k$. We say that a set is totally disconnected if it has no connected subsets containing more than one point.

Problem 2 (ii). *Is K totally disconnected?*

The negative answer to Problem 2 (ii) and a soft argument would yield the negative answer to Problem 2 (i). It is easy to see that for any fixed $t \in [0,1]$, $X_t \in K$, a.s. Hence, the dimension of K is equal to 2, a.s. The problem is related to the existence of "cut points;" see Burdzy (1989, 1995). It is also related to the question of whether $X[0,1]$ is a "universal planar curve" or equivalently, whether it contains a homeomorphic image of the Sierpiński carpet; see Mandelbrot (1982, Section VIII.25).

3. Percolation dimension of planar Brownian trace. Let $\dim(A)$ denote the Hausdorff dimension of a set A. The *percolation dimension* of a set B is the infimum of $\dim A$, where the infimum is taken over all Jordan arcs $A \subset B$ which contain at least two distinct points. Suppose that X_t is a two-dimensional Brownian motion.

Problem 3. *Is the percolation dimension of $X[0,1]$ equal to 1?*

Related paper: Burdzy (1990b).

4. Efficient couplings in acute triangles. Suppose that D is a triangle with all angles strictly smaller than $\pi/2$ and let $\mu_2 > 0$ be the second eiganvalue for the Laplacian in D with Neumann boundary conditions. The first eigenvalue is zero.

Problem 4. *Can one construct two reflected Brownian motions X_t and Y_t in D starting from different points and such that $\tau = \inf\{t \geq 0 : X_t = Y_t\} < \infty$ a.s., and for every fixed $\varepsilon > 0$,*

$$P(\tau > t) \leq \exp(-(\mu_2 - \varepsilon)t),$$

for large t?

See Burdzy and Kendall (2000) for the background of the problem.

5. Convergence of synchronous couplings. Suppose $D \subset \mathbf{R}^2$ is an open connected set with smooth boundary, not necessarily simply connected. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D$ and suppose that $x_0, y_0 \in \overline{D}$. Let B be standard planar Brownian motion and consider processes X and Y solving the following equations,

$$X_t = x_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X \qquad \text{for } t \geq 0,$$

$$Y_t = y_0 + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y \qquad \text{for } t \geq 0.$$

Here L^X is the local time of X on ∂D, i.e, it is a non-decreasing continuous process which does not increase when X is in D. In other words, $\int_0^\infty \mathbf{1}_D(X_t) dL_t^X = 0$, a.s. The same remarks apply to L^Y. We call (X, Y) a "synchronous coupling."

Problem 5. *(i) Does there exist a bounded planar domain such that with positive probability,* $\limsup_{t\to\infty} |X_t - Y_t| > 0$*?*

(ii) If D is the complement of a non-degenerate closed disc, is it true that with positive probability, $\limsup_{t\to\infty} |X_t - Y_t| > 0$*?*

If there exists a bounded domain D satisfying the condition in Problem 5 (i) then it must have at least two holes, by the results in Burdzy, Chen and Jones (2006). See that paper and Burdzy and Chen (2002) for the background of the problem.

6. Non-extinction of a Fleming-Viot particle model. Consider a branching particle system $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$ in which individual particles X^j move as N independent Brownian motions and die when they hit the complement of a fixed domain $D \subset \mathbf{R}^d$. To keep the population size constant, whenever any particle X^j dies, another one is chosen uniformly from all particles inside D, and the chosen particle branches into two particles. Alternatively, the death/branching event can be viewed as a jump of the j-th particle.

Let τ_k be the time of the k-th jump of $\mathbf{X}_t$. Since the distribution of the hitting time of ∂D by Brownian motion has a continuous density, only one particle can hit ∂D at time τ_k, for every k, a.s. The construction of the process is elementary for all $t < \tau_\infty = \lim_{k\to\infty} \tau_k$. However, there is no obvious way to continue the process $\mathbf{X}_t$ after the time τ_∞ if $\tau_\infty < \infty$. Hence, the question of the finiteness of τ_∞ is interesting. Theorem 1.1 in Burdzy, Hołyst and March (2000) asserts that $\tau_\infty = \infty$, a.s., for every domain D. Unfortunately, the proof of that theorem contains an irreparable error. It has been shown in Bieniek, Burdzy and Finch (2009) that $\tau_\infty = \infty$, a.s., if the domain $D \subset \mathbf{R}^d$ is Lipschitz with a Lipschitz constant depending on d and the number N of particles.

Problem 6. *Is it true that* $\tau_\infty = \infty$*, a.s., for any bounded open connected set* $D \subset \mathbf{R}^d$*?*

7. Are shy couplings necessarily rigid? (Proposed by K. Burdzy and W. Kendall).

Suppose that $D \subset \mathbf{R}^d$, $d \geq 2$, is a bounded connected open set and let X_t and Y_t be reflected Brownian motions in D defined on the same probability space.

Problem 7. *Suppose that there exist reflected Brownian motions X_t and Y_t in D and $\varepsilon > 0$ such that $\inf_{t\geq 0} |X_t - Y_t| \geq \varepsilon$ with probability greater than 0. Does this imply that there exist reflected Brownian motions X'_t and Y'_t in D, $\varepsilon > 0$ and a deterministic function f such that $f(X'_t) = Y'_t$ for all $t \geq 0$, a.s., and $\inf_{t\geq 0} |X'_t - Y'_t| \geq \varepsilon$ with probability greater than 0?*

Example 3.9 of Benjamini, Burdzy and Chen (2007) shows that there exists a graph Γ and Brownian motions X_t and Y_t on Γ such that $\inf_{t\geq 0} |X_t - Y_t| \geq \varepsilon$ with probability greater than 0 but Y_t is not a deterministic function of X_t. Moreover, all bijective isometries of Γ have fixed points.

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