



Archimedes' Principle for Ideal Gas

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Received: 4 February 2021 / Accepted: 12 February 2022

Published online: 14 March 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract: We prove Archimedes' principle for a macroscopic ball in ideal gas consisting of point particles with non-zero mass. The main result is an asymptotic theorem, as the number of point particles goes to infinity and their total mass remains constant. We also show that, asymptotically, the gas has an exponential density as a function of height. We find the asymptotic inverse temperature of the gas. We derive an accurate estimate of the volume of the phase space using the local central limit theorem.

1. Introduction

There seems to be no rigorous proof of Archimedes' principle in the mathematical literature. The most likely reason for this omission is that Archimedes' principle is trivial given a few natural assumptions. The principle can be easily derived using the divergence theorem, assuming that the formula for the pressure as a function of height is known. The “barometric formula” which says that the pressure in gas has an exponential density as a function of height can be easily derived from the ideal gas law. While this derivation of Archimedes' principle is sufficient for the scientific applications, one could ask whether the principle can be derived from a more fundamental model of the matter, as in Hilbert's 6-th problem. This is what we will do in the present article. Perhaps the most significant difference between our model and the derivation of Archimedes' principle alluded to above is that the floating object is allowed to move in our case. We are not aware of an existing proof of Archimedes' principle, rigorous or not, based on a model with a moving floating object.

We will consider a container with a bounded base, vertical side walls and no top. The container will hold point particles (ideal gas) and a floating object in the shape of a ball and no internal structure (the mass will be uniformly spread over the ball). The spherical

KB's research was supported in part by Simons Foundation Grant 506732. J. Małecki was supported by the Polish National Science Centre (NCN) grant no. 2018/29/B/ST1/02030.

shape of the floating object allows us to avoid the discussion of the energy going into rotation—the collisions of the ball with point particles and the walls of the container will not induce rotation of the ball.

The point particles and the ball will move according to Newton’s laws in a gravitational field with constant acceleration. We will assume conservation of energy and momentum but this assumption does not have a unique interpretation in the case of collisions of point particles with the infinitely heavy walls of the container. We will consider two types of reflections of point particles from the walls of the container: (i) specular reflections where the angle of reflection is equal to the angle of incidence, and (ii) random reflections according to the Lambertian distribution also known as the Knudsen law. We will assume that the system is in equilibrium so that its density is given by the microcanonical ensemble formula. We will prove that this distribution is the unique invariant measure in case (ii). Simple examples show that there are multiple invariant measures if we assume specular reflections.

In our asymptotic results we will assume that the following objects and quantities are fixed: the container, the mass and radius of the macroscopic ball, the total mass of the gas (all point particles), the total (potential and kinetic) energy of all moving objects (point particles and the ball), and the gravitational acceleration. The number of point particles will go to infinity so the mass of a single particle will go to zero.

On the way to the main result, Archimedes’ principle, we will derive a few other results that may have independent interest.

We will present an accurate formula for the volume of the phase space. Our calculation is based on the local central limit theorem. This theorem was proved long time ago but the literature is hard to follow so we hope that our short review of that literature will help those readers who might need this result in their own research.

The microcanonical ensemble formula is well known, see, e.g., [Rue99, Sect. 1.2]. We could not find a version of the formula needed here in the literature so a derivation was supplied in a parallel project [BDG+21]. We will not reproduce that proof here but we will present a brief review.

We will prove a version of the barometric formula, i.e., we will show that the density of the gas has, asymptotically, an exponential density as a function of height. We will show that the parameter of the exponential distribution can be identified with the inverse temperature.

1.1. Literature review. A version of Archimedes’ principle was proved in [BCP11] but that model was completely different from the present one. The “gas” consisted of hard spheres with strictly positive radius. Their centers moved according to independent Brownian motions (except for the collisions). The number of “gas particles” was constant and the asymptotic theorems were proved by sending the “gravitational acceleration” to infinity.

Our article is concerned with a model in which a macroscopic object interacts with microscopic particles according to Newtonian mechanics. In this sense, our model is closest to the “piston problem” proposed in [Lie99]. A large number of papers were inspired by [Lie99] and devoted to the piston problem; see, for example, [Sin99, LPS00, CL02, CLS02, LSC02, NS04, Gor11, IS15] and references therein. Several different models were considered in those papers. In one of the models, a piston moves along a tube and is bombarded by microscopic particles from both sides.

1.2. Limitations of the model. Our model is, obviously, an oversimplified representation of reality and there is no hope that it could be modified to be very realistic. Still, from the mathematical point of view, some aspects of the model might be worth generalizing in future research.

- (i) In our model, the macroscopic ball has no internal structure so it has negligibly small heat capacity. It might be possible to analyze a model in which the ball is replaced with a “balloon,” i.e., an infinitely thin sphere holding inside (mobile) point particles with different masses than those of the outside particles.
- (ii) Since we assume that the gas is ideal, the point particles do not interact and, therefore, their collisions with the macroscopic ball are the only way in which they can exchange energy between each other. This is the only way in which the energy may become approximately equidistributed in the stationary regime. To generalize our results to gases consisting of hard spheres with positive radius one would need accurate estimates of the volume of the phase space. The virial expansion might be useful in this context, see [MM77, Sect. 8i] or [Rue99, Sect. 4.3]. If this approach works, it will require a whole new set of calculations.
- (iii) In our model, the container has a flat bottom and vertical side walls. Many of our calculations depend on this assumption.

1.3. Organization of the paper. The rigorous presentation of the model and the statement of our main results are in Sect. 2. We will review known results on the local central limit theorem in Sect. 3. Section 4 contains a review of the results on the microcanonical ensemble formula developed in a parallel paper. We will derive an accurate estimate of the phase space volume in Sect. 5. The proof of Archimedes' principle will be given in Sect. 6. The inverse temperature will be identified in Sect. 7. The uniqueness of the stationary probability distribution under Lambertian reflections will be proved in Sect. 8.

2. Model and Main Results

We will consider n point particles (ideal gas) and one macroscopic hard ball in a d -dimensional container D with vertical walls, a bounded base, extending upward to infinity in the vertical direction.

Suppose that $d \geq 2$ and let $D_b \subset \mathbb{R}^{d-1}$ denote the bottom of the container $D = D_b \times [0, \infty)$ in $\mathbb{R}^d$. The radius of the $(n+1)$ -st (macroscopic) ball will be fixed and denoted $R > 0$. Obviously, the point particles will have radii equal to 0. We will assume that D_b has a smooth boundary and satisfies the inner ball condition with radius R , i.e., for every point $x \in \partial D_b$, there is a unique $(d-1)$ -dimensional open ball with radius R inside D_b whose boundary is tangent to ∂D_b at x and, moreover, x is the only point in the intersection of the ball boundary and ∂D_b . We will also assume that a closed ball of radius R fits in the interior of D , so that each point particle has room to move from below to above the ball, and vice versa.

We will assume that the mass of the i th point particle is $m_i = m/n > 0$ for $i = 1, \dots, n$, for some $m > 0$. When we let $n \rightarrow \infty$ in our theorems, the total mass of point particles will remain constant and equal to m . The mass of the macroscopic ball will be denoted $M = m_{n+1} > 0$. We will assume that the mass is evenly spread over the volume of the macroscopic ball. The walls of the container will be assumed to have infinite mass—this assumption is a way of specifying the meaning of “totally elastic collisions” of point particles and the ball with the walls of the container.

We will assume that the point particles and the ball are moving within a gravitational field with the constant acceleration $g > 0$ pointing downwards. The point particles will not interact with each other. They will reflect from the walls of the container and they will undergo totally elastic collisions with the ball. The collisions of the ball with the walls of the container will be totally elastic.

Time will be suppressed in the notation, except for Sect. 8. We will assume that the system is in equilibrium. Random objects will represent the state of the system at time 0 (or any other fixed time).

Let $(X_i, Y_i) \in \mathbb{R}^d$ denote the random position of the i th point particle or the center of the macroscopic ball (for $i = n + 1$), where $X_i \in \mathbb{R}^{d-1}$ represents the horizontal coordinates and $Y_i \geq 0$ is the vertical coordinate ($Y_{n+1} \geq R$). By $V_i \in \mathbb{R}^d$ we will denote the random velocity of the i -th point particle or the ball.

Let

$$\mathbf{x}_k = (x_1, \dots, x_k), \quad \mathbf{y}_k = (y_1, \dots, y_k), \quad \mathbf{v}_k = (v_1, \dots, v_k), \quad (2.1)$$

$$\mathbf{X}_k = (X_1, \dots, X_k), \quad \mathbf{Y}_k = (Y_1, \dots, Y_k), \quad \mathbf{V}_k = (V_1, \dots, V_k), \quad (2.2)$$

where $x_i \in \mathbb{R}^{d-1}$, $y_i \in \mathbb{R}_+$, and $v_i \in \mathbb{R}^d$. In the notation given above, upper case letters represent random variables and lower case letters represent their values.

We will consider $(\mathbf{X}_{n+1}, \mathbf{Y}_{n+1}, \mathbf{V}_{n+1})$ to be a random vector distributed according to the microcanonical ensemble formula (2.7) stated below, although we will give different distributions to these random vectors in some of the proofs.

We will assume that the total energy of our system, E , is fixed. Hence, a.s.,

$$E = \sum_{i=1}^{n+1} \left(m_i g Y_i + \frac{m_i \|V_i\|^2}{2} \right), \quad (2.3)$$

with the convention that the zero level represents zero potential energy. We will always assume that $E > MgR$ so that the ball and point particles cannot rest motionless at the bottom.

A ball in $\mathbb{R}^d$ with center (x, y) and radius r will be denoted $\mathcal{B}((x, y), r)$. Let

$$D'_b = \{x \in D_b : \mathcal{B}((x, y), R) \subset D \text{ for all } y > R\}, \quad (2.4)$$

$$\mathbf{D}_n = \left\{ (\mathbf{x}_n, x_{n+1}, \mathbf{y}_n, y_{n+1}, \mathbf{v}_{n+1}) \in D_b^n \times D'_b \times \mathbb{R}_+^n \times [R, \infty) \times \mathbb{R}^{nd} : \right. \\ \left. (x_k, y_k) \in D \setminus \mathcal{B}((x_{n+1}, y_{n+1}), R), \quad k = 1, \dots, n \right\}, \quad (2.5)$$

$$\mathbf{D}_n^E = \left\{ (\mathbf{x}_{n+1}, \mathbf{y}_{n+1}, \mathbf{v}_{n+1}) \in \mathbf{D}_n : \sum_{i=1}^{n+1} \left(m_i y_i g + \frac{1}{2} m_i v_i^2 \right) = E \right\}.$$

$$\mathbf{D}_n^E(y_{n+1}) = \left\{ (\mathbf{x}_n, x_{n+1}, \mathbf{y}_n) \in D_b^n \times D'_b \times \mathbb{R}_+^n : \sum_{i=1}^{n+1} m_i y_i g \leq E, \right. \\ \left. (x_k, y_k) \in D \setminus \mathcal{B}((x_{n+1}, y_{n+1}), R), \quad k = 1, \dots, n \right\}. \quad (2.6)$$

Let $\mu_{\mathbf{y}_{n+1}}(ds)$ denote the uniform probability measure on the sphere in $\mathbb{R}^{(n+1)d}$ (so that the sphere is $((n+1)d - 1)$ -dimensional), centered at the origin, with the radius $\left(2E - 2 \sum_{i=1}^{n+1} m_i y_i g\right)^{1/2}$. Consider the following measure $\mathbb{P}_n$ on $\mathbf{D}_n^E$,

$$\mathbb{P}_n(d\mathbf{x}_{n+1} d\mathbf{y}_{n+1} d\mathbf{v}_{n+1}) = C \left(E - \sum_{i=1}^{n+1} m_i y_i g \right)^{((n+1)d-2)/2} \mu_{\mathbf{y}_{n+1}} \left(\frac{dv_1}{\sqrt{m_1}}, \dots, \frac{dv_{n+1}}{\sqrt{m_{n+1}}} \right) d\mathbf{x}_{n+1} d\mathbf{y}_{n+1}, \quad (2.7)$$

where C is the normalizing constant so that $\mathbb{P}_n$ is a probability measure. The measure $\mathbb{P}_n$ is a special case of the “microcanonical ensemble formula,” see, e.g., [Rue99, Sect. 1.2].

We will say that a point particle undergoes a Lambertian reflection from a surface or that the particle reflects according to the Knudsen law if the reflection occurs at a point where the inner normal to the surface is uniquely defined, the probability density of the angle between the reflected trajectory and the normal vector is proportional to the cosine of the angle, and the law is invariant under rotations about the normal vector. Note that the Lambertian distribution of the outgoing velocity vector is independent from the incoming velocity vector, except that the two velocities have the same norm, so that energy is conserved. This law for random reflections was considered by Lambert in the context of light reflection from rough surfaces ([Lam60]) and by Knudsen ([Knu34]) as a model for gas particle reflections. By [BDG+21, Cor. 3.2] (which can be derived from [Pla12, Thm. 4.1] or [ABS13, Thm. 2.2]), the Lambertian reflection law is the unique random reflection law that does not depend on the angle of incidence and is consistent with the standard (specular) reflection law from a rough surface consisting of small crystals with smooth reflecting surfaces.

Theorem 2.1. (i) *The measure $\mathbb{P}_n$ (microcanonical ensemble formula) is invariant for the dynamical system defined above and two types of reflections:*

- (a) *totally elastic reflections between any pair of objects,*
- (b) *independent Lambertian reflections for point particles reflecting from the container walls (including the bottom) and totally elastic reflections between any other pair of objects.*

(ii) *In case (b), $\mathbb{P}_n$ is the unique non-degenerate stationary probability distribution for the system of point particles and the macroscopic ball. The following are the only classes of “degenerate” invariant distributions:*

- (1) *Invariant distributions such that no point particle ever hits a wall of the container.*
- (2) *Invariant distributions such that at least one point particle has no energy, so that it is resting at the bottom of the container.*

The proof of Theorem 2.1 will be given in Sects. 4 and 8.

Remark 2.2. (i) Part (ii) of Theorem 2.1 is not true under assumption (a). There are many elementary examples of invariant measures for our dynamical system if we assume specular reflections of point particles from the walls of the container. To construct one of them, place all point particles on disjoint vertical lines which do not intersect the macroscopic ball at time 0. Make initial velocities vertical for the ball and all point particles. Each of these objects will stay on a vertical line forever. To construct an invariant measure, use the ergodic theorem.

- (ii) We call invariant distributions in (1) degenerate because Lambertian reflections are never activated and the system has no opportunity to mix. The following example

of an invariant distribution illustrates family (1). Suppose that the macroscopic ball has a vertical initial velocity and all point particles are located on the vertical line passing through the ball's center, they are placed above the ball, and they all have vertical initial velocities. In this case the center of the ball and the point particles will remain on the same vertical line forever and their velocities will also remain vertical (although the positions and velocities will not remain constant). For this initial condition, point particles will not hit the walls of the container so there will be no opportunity for the random reflections to cause mixing in the system. Just like in part (i) of the remark, one can use the ergodic theorem to construct an invariant measure.

(iii) Invariant distributions in (2) are degenerate because in this case the number of point particles is effectively less than n .

We will often use integrals of the form $\int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \lambda e^{-\lambda r} dx dr$. In cases like this (and similar), $\tilde{x}$ should be interpreted as any point in D'_b , dx will represent Lebesgue measure in $\mathbb{R}^{d-1}$, and dr will represent Lebesgue measure in $\mathbb{R}$. Note that the integral does not depend on $\tilde{x}$ as long as $\tilde{x} \in D'_b$.

In the following theorem we present the macroscopic equations for the density of the gas and the vertical position of the ball.

Theorem 2.3. Fix d, D, R, M, m, g and E . Consider the equations

$$M = m \frac{\int_{\mathcal{B}((\tilde{x}, y), R)} \lambda e^{-\lambda r} dx dr}{\int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \lambda e^{-\lambda r} dx dr}, \quad (2.8)$$

$$\frac{dmg}{2\lambda} + mg \frac{\int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} r \lambda e^{-\lambda r} dx dr}{\int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \lambda e^{-\lambda r} dx dr} + Mgy = E, \quad (2.9)$$

with unknowns $y \geq R$ and $\lambda > 0$.

(i) There exists a unique λ_* which satisfies (2.9) with $y = R$.

(ii) Suppose that

$$M < m \frac{\int_{\mathcal{B}((\tilde{x}, R), R)} \lambda_* e^{-\lambda_* r} dx dr}{\int_{D \setminus \mathcal{B}((\tilde{x}, R), R)} \lambda_* e^{-\lambda_* r} dx dr}. \quad (2.10)$$

Then (2.8)–(2.9) have a unique solution that will be denoted (y_A, λ_A) . Moreover $R < y_A < E/(Mg)$.

The proof of Theorem 2.3 will be given in Sect. 5.

For $\tilde{x} \in D'_b$, $y' \geq R$ and $\lambda > 0$, let $\nu_{\tilde{x}, y', \lambda}$ be the probability distribution defined by

$$\nu_{\tilde{x}, y', \lambda}(dx, dy) = \frac{\mathbf{1}_{D \setminus \mathcal{B}((\tilde{x}, y'), R)}(x, y) \lambda e^{-\lambda y}}{\int_{D \setminus \mathcal{B}((\tilde{x}, y'), R)} \lambda e^{-\lambda r} dz dr} dx dy, \quad x \in D_b, y > 0. \quad (2.11)$$

Let $\delta_{(X_i, Y_i)}$ denote the probability measure on D which consists of a single atom at (X_i, Y_i) . The normalized (probability) empirical distribution $\mathbb{Q}_n$ of the gas (point particles) is defined as

$$\mathbb{Q}_n = \frac{1}{n} \sum_{i=1}^n \delta_{(X_i, Y_i)}. \quad (2.12)$$

The following theorem is our main result. We do not specify the type of scattering in the theorem because the statement is based on the distribution $\mathbb{P}_n$ (microcanonical ensemble formula) and, hence, it applies to each of the two types of scattering discussed in Theorem 2.1.

Theorem 2.4. (Archimedes' principle) *Fix d, D, R, M, m, g and E . Assume that the distribution of $(\mathbf{X}_{n+1}, \mathbf{Y}_{n+1}, \mathbf{V}_{n+1})$ is $\mathbb{P}_n$. Recall the notation from Theorem 2.3 and assume that (2.10) holds.*

(i) *For every $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n(|Y_{n+1} - y_A| > \varepsilon) = 0. \quad (2.13)$$

(ii) *The marginal distribution of X_{n+1} under $\mathbb{P}_n$ is uniform in D'_b . Given $\{X_{n+1} = x\}$, the conditional distribution of $\mathbb{Q}_n$ converges to ν_{x, y_A, λ_A} weakly, in probability as $n \rightarrow \infty$.*

The proof of Theorem 2.4 will be given in Sect. 6.

Remark 2.5. (i) Theorem 2.4 is a mathematical representation of Archimedes' principle. Part (ii) of the theorem identifies the limiting empirical distribution of gas particles when $n \rightarrow \infty$. The gas density is constant on the horizontal hyperplanes and it is exponential as a function of the height, outside the ball.

Part (i) of Theorem 2.4 and (2.8) say that the ball is likely to float at the height such that the weight of the ball is equal to the weight of the displaced gas (i.e., (2.8) holds), assuming that the gas is distributed as in part (ii).

(ii) We need an extra Eq. (2.9) to identify y_A and λ_A uniquely. The equation is an expression of additivity and conservation of energy. The three terms on the left hand side represent (asymptotically) the kinetic energy of the gas, the potential energy of the gas, and the potential energy of the ball. The kinetic energy of the ball is negligibly small asymptotically.

The form of the first term on the left hand side of (2.9), i.e., $dmg/(2\lambda_A)$, representing the kinetic energy of the gas, is a manifestation of the virial theorem, which states how the energy is distributed between potential and kinetic forms. The virial theorem is by now a classical result, discovered by Clausius in 1870 (see, for example [Col78]).

Formula (2.9) reduces to

$$\frac{dmg}{2\lambda_A} + \frac{mg}{\lambda_A} = E$$

in the case when there is no ball, i.e., $R = 0$ and $M = 0$. In other words, the ratio of potential to kinetic energy is $2/d$ for pure gas.

Alternatively one could say that $dmg/(2\lambda_A)$ represents the kinetic energy, hence the heat energy, because λ_A can be identified with the inverse temperature (see (7.1) for a precise formula).

(iii) The heuristic meaning of the first claim of Theorem 2.3 is that there is a unique asymptotic distribution of gas if the ball rests at the bottom of the container. The second part says that if the weight of the ball is smaller than the weight of the gas displaced by the ball when it is placed at the bottom then, asymptotically, there is a unique level at which the ball will float and the corresponding unique distribution of the gas.

3. Local Central Limit Theorem

For any random variable A , let f_A denote the density of A (if it exists). Suppose that random variables ξ_k , $k \geq 1$, are i.i.d. with $\mathbb{E}\xi_k = 0$, $E\xi_k^2 = 1$ and density f_{ξ_k} . Let $M_3 = \mathbb{E}|\xi_k|^3$ and $S_n = \sum_{k=1}^n \xi_k / (\sqrt{n}\sigma)$. Let $\varphi(x)$ denote the standard normal density, i.e., $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

The following result is Theorem 1 in [Sv65].

Lemma 3.1. *There exists an absolute constant C such that if $f_{\xi_k}(x) \leq C_1$ for all $x \in \mathbb{R}$ then for all $n \geq 1$,*

$$\sup_{x \in \mathbb{R}} |f_{S_n}(x) - \varphi(x)| \leq \frac{CM_3^2 \max(1, C_1^3)}{\sqrt{n}}. \quad (3.1)$$

Remark 3.2. There is a considerable literature on the local central limit theorem but the results are hard to extract from that literature for a number of reasons, including absence of some journals from accessible libraries and lack of proofs in a number of publications.

The bound in Lemma 3.1 is given in Remark 5, Section 4, Chapter VII in [Pet75]. The bound is given there without a proof and it is attributed to [Š66]. Unfortunately, [Š66] does not contain a proof. A similar bound is a special case of Theorem 1 in [Š71] but, once again, that paper does not contain a proof.

One can derive Lemma 3.1 from Theorem 4 in [Sta65].

The book [Pet75] contains several versions of the local central limit theorem, see, e.g., Theorem 15 in Chapter VII. However, in each of these theorems the error is given in the form $o(1/\sqrt{n})$. One could extract a bound of the form $c/\sqrt{n}$ with an explicit formula for c depending on the moments of the summands from the proof but that would be a very tedious task.

4. Microcanonical Ensemble Formula

Proof of Theorem 2.1 (i). The formula (2.7) is a version of a well known “microcanonical ensemble formula,” see, e.g., [Rue99, Sect. 1.2]. We could not find a rigorous proof that this distribution is invariant in the contemporary literature. A proof has been given in [BDG+21], a parallel project. The result proved in [BDG+21, Prop. 5.3] implies that

$$f(\mathbf{x}_{n+1}, \mathbf{y}_{n+1}, \mathbf{v}_{n+1}) := \left(E - \sum_{i=1}^{n+1} m_i y_i g \right)^{((n+1)d-2)/2} \frac{d\mu_{\mathbf{y}_{n+1}}}{d\mathbf{v}_{n+1}} \left(\frac{v_1}{\sqrt{m_1}}, \dots, \frac{v_{n+1}}{\sqrt{m_{n+1}}} \right)$$

is the density of an invariant measure on the whole space $\mathbb{R}^{2(n+1)d}$.

We claim that this density restricted to $\mathbf{D}_n$ is invariant for our dynamical system with reflections. It suffices to show that the specular and Lambertian reflections between the point particles, the ball and the walls of the container leave the measure invariant. For the specular reflections, the proof is essentially identical to that of the proof of [BDG+21, Prop. 2.3]. The result can be extended to Lambertian reflections of point particles from the wall, as shown in [BDG+21, Prop. 3.3, Remark 3.4] (see the remarks preceding and following that result). $\square$

5. Phase Space Volume

We will use two different sets of notations for some quantities. While this may create some confusion, we will explain why this convention has some advantages.

For $n = 0, 1, 2, \dots$, $\gamma > 0$ and $y \geq R$ we let

$$\hbar_n = \hbar_n(\gamma, y) = \int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \gamma r^n e^{-\gamma r} dx dr, \quad (5.1)$$

where $\tilde{x} \in D'_b$. The definition of and notation for $\hbar_n$ lead to the following very simple formula, obtained using the dominated convergence theorem,

$$\frac{\partial \hbar_n}{\partial \gamma} = \frac{\hbar_n}{\gamma} - \hbar_{n+1}. \quad (5.2)$$

Some of the functions $\hbar_n$ have physical meaning, so we find it easier to memorize them if different notation is used in some cases. Specifically,

$$q(\gamma, y) = \int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \gamma e^{-\gamma r} dx dr = \hbar_0(\gamma, y), \quad (5.3)$$

$$u(\gamma, y) = \frac{1}{q(\gamma, y)} \int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} r \gamma e^{-\gamma r} dx dr = \frac{\hbar_1(\gamma, y)}{\hbar_0(\gamma, y)}, \quad (5.4)$$

$$w(\gamma, y) = \int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \gamma r^2 e^{-\gamma r} dx dr = \hbar_2(\gamma, y). \quad (5.5)$$

We will prove in Lemma 5.2 that for given $u > 0$ and $y \geq R$, the equation $u = u(\lambda, y)$ uniquely defines $\lambda = \lambda(u, y)$. We let

$$h_n(u, y) = \hbar_n(\lambda(u, y), y), \quad q(u, y) = q(\lambda(u, y), y), \quad w(u, y) = w(\lambda(u, y), y). \quad (5.6)$$

Lemma 5.1. For all $y' \geq R$ and $\gamma > 0$,

$$\frac{1}{8}(1 - e^{-1/2}) \leq \frac{q(\gamma, y')}{|D_b|} \leq 1, \quad (5.7)$$

$$\frac{1}{200} \leq \gamma u(\gamma, y') \leq \frac{8}{1 - e^{-1/2}} < 21, \quad (5.8)$$

$$2^{-12}e^{-1} \leq \gamma^2 \left(\frac{w(\gamma, y')}{q(\gamma, y')} - u(\gamma, y')^2 \right) \leq \frac{16}{1 - e^{-1/2}}, \quad (5.9)$$

$$2^{-21} < \frac{1 - e^{-1/2}}{16} \frac{1}{200^2} \leq \frac{u(\gamma, y')^2}{w(\gamma, y')/q(\gamma, y') - u(\gamma, y')^2} \leq \frac{2^{18}e}{(1 - e^{-1/2})^2} < 2^{23}. \quad (5.10)$$

Proof. Let $\mathcal{B}_* = \mathcal{B}((\tilde{x}, y), R)$. Elementary geometry shows that if

$$A = \left\{ x \in D_b : |x - \tilde{x}| < \frac{\sqrt{3}}{2}R \right\}$$

then $(D_b \setminus A) \times (0, y' - R/2) \in D \setminus \mathcal{B}_*$. If $\sigma(r)$ denotes the volume of a $(d-1)$ -dimensional ball then, using our assumption that $d \geq 2$,

$$\frac{|D_b \setminus A|}{|D_b|} = 1 - \frac{|A|}{|D_b|} \geq 1 - \frac{\sigma(\sqrt{3}R/2)}{\sigma(R)} = 1 - \left(\frac{\sqrt{3}}{2}\right)^{d-1} \geq 1 - \frac{\sqrt{3}}{2} > \frac{1}{8}. \quad (5.11)$$

We have

$$q = \int_{D \setminus \mathcal{B}_*} \gamma e^{-\gamma r} dx dr \geq \int_{D_b} dx \int_{y'+R}^{\infty} \gamma e^{-\gamma r} dr = |D_b| \exp(-\gamma(y' + R)). \quad (5.12)$$

By (5.11),

$$\begin{aligned} q &= \int_{D \setminus \mathcal{B}_*} \gamma e^{-\gamma r} dx dr \geq \int_{D_b \setminus A} dx \int_0^{y'-R/2} \gamma e^{-\gamma r} dr \\ &= |D_b \setminus A| (1 - \exp(-\gamma(y' - R/2))) \\ &\geq \frac{1}{8} |D_b| (1 - \exp(-\gamma(y' - R/2))). \end{aligned} \quad (5.13)$$

Recall that $y' \geq R$. If $\gamma \leq 1/y'$ then (5.12) yields

$$\begin{aligned} q &\geq |D_b| \exp(-\gamma(y' + R)) \geq |D_b| \exp(-(1/y')(y' + R)) = |D_b| \exp(-1 - R/y') \\ &\geq |D_b| e^{-2}. \end{aligned} \quad (5.14)$$

In the case when $\gamma \geq 1/y'$, we use (5.13) to see that

$$\begin{aligned} q &\geq \frac{1}{8} |D_b| (1 - \exp(-\gamma(y' - R/2))) \geq \frac{1}{8} |D_b| (1 - \exp(-(1/y')(y' - R/2))) \\ &= \frac{1}{8} |D_b| (1 - \exp(-1 + R/(2y'))) \geq \frac{1}{8} |D_b| (1 - \exp(-1 + 1/2)) \\ &= \frac{1}{8} |D_b| (1 - e^{-1/2}). \end{aligned}$$

Since $e^{-2} > \frac{1}{8}(1 - e^{-1/2})$, the above estimate and (5.14) give

$$q \geq \frac{1}{8} |D_b| (1 - e^{-1/2}). \quad (5.15)$$

This proves the lower bound in (5.7). The upper bound follows directly from the definition of q .

We use (5.15) to derive the upper bound in (5.8) as follows,

$$\begin{aligned} u &= \frac{\int_{D \setminus \mathcal{B}_*} r \gamma e^{-\gamma r} dx dr}{\int_{D \setminus \mathcal{B}_*} \gamma e^{-\gamma r} dx dr} = \frac{1}{q} \int_{D \setminus \mathcal{B}_*} r \gamma e^{-\gamma r} dx dr \leq \frac{1}{q} \int_D r \gamma e^{-\gamma r} dx dr \\ &= \frac{1}{q} |D_b| \frac{1}{\gamma} \leq \frac{8}{(1 - e^{-1/2})\gamma}. \end{aligned} \quad (5.16)$$

Let $W = (W_1, W_2)$ be a random vector with the distribution $\nu = \nu_{\tilde{x}, y', \gamma}$ defined in (2.11), where $W_1 \in D_b$ and $W_2 > 0$. Note that

$$w/q - u^2 = \text{Var } W_2. \quad (5.17)$$

In view of (5.11) and (5.7), the marginal density $f_{W_2}(y)$ satisfies

$$f_{W_2}(y) \leq \frac{|D_b|}{q} \gamma e^{-\gamma y} \leq \frac{8}{1 - e^{-1/2}} \gamma e^{-\gamma y}, \quad y > 0, \quad (5.18)$$

$$f_{W_2}(y) \geq \gamma e^{-\gamma y}, \quad y \geq y' + R, \quad (5.19)$$

$$f_{W_2}(y) \geq \frac{1}{8} \gamma e^{-\gamma y}, \quad y \leq y' - R/2. \quad (5.20)$$

It follows from (5.18) that

$$\begin{aligned} w/q - u^2 = \text{Var } W_2 &\leq \mathbb{E} W_2^2 = \int_0^\infty y^2 f_{W_2}(y) \, dy \leq \int_0^\infty y^2 \frac{8}{1 - e^{-1/2}} \gamma e^{-\gamma y} \, dy \\ &= \frac{8}{1 - e^{-1/2}} \frac{2}{\gamma^2}. \end{aligned}$$

This gives the upper bound in (5.9).

Suppose that $1/\gamma \leq y'$ and let $y'' = 1/\gamma$. Since $y' \geq R$ and $y'' \leq y'$, we have $y''/2 \leq y' - R/2$. Hence, we can apply (5.20) to $y \in (0, y''/2)$ and obtain

$$\begin{aligned} u = \mathbb{E} W_2 &\geq \int_0^{y''/2} y f_{W_2}(y) \, dy \geq \int_0^{y''/2} y \frac{1}{8} \gamma e^{-\gamma y} \, dy \\ &= \frac{1}{8\gamma} (1 - e^{-y''\gamma/2} (y''\gamma/2 + 1)) = \frac{1}{8\gamma} (1 - (3/2)e^{-1/2}) > \frac{1}{200\gamma}. \end{aligned} \quad (5.21)$$

Suppose that $1/\gamma \geq y'$ and let $y'' = 1/\gamma$. Since $y' \geq R$ and $y'' \geq y'$, we have $2y'' \geq y' + R$. Hence, we can apply (5.19) to $y > 2y''$ and obtain

$$\begin{aligned} u = \mathbb{E} W_2 &\geq \int_{2y''}^\infty y f_{W_2}(y) \, dy \geq \int_{2y''}^\infty y \gamma e^{-\gamma y} \, dy = \frac{1}{\gamma} e^{-2y''\gamma} (2y''\gamma + 1) \\ &= \frac{1}{\gamma} 3e^{-2} > \frac{1}{3\gamma}. \end{aligned}$$

This and (5.21) yield the lower bound in (5.8).

Suppose that $1/\gamma \leq y'$ and let $y'' = 1/\gamma$. Then for any $a_0 \in \mathbb{R}$ there exists an interval $(a_1, a_2) \subset (0, y''/2)$ such that $a_2 - a_1 \geq y''/8$ and for any $y \in (a_1, a_2)$, we have $|y - a_0| \geq y''/8$. We will apply this observation with $a_0 = \mathbb{E} W_2$. Since $y' \geq R$ and $y'' \leq y'$, we have $a_2 \leq y''/2 \leq y' - R/2$. Hence, we can apply (5.20) to $y \in (a_1, a_2)$ and obtain

$$\begin{aligned} \text{Var } W_2 &\geq \int_{a_1}^{a_2} |y - \mathbb{E} W_2|^2 f_{W_2}(y) \, dy \geq \int_{a_1}^{a_2} (y''/8)^2 \frac{1}{8} \gamma e^{-\gamma y} \, dy \\ &\geq (a_2 - a_1)(y''/8)^2 \frac{1}{8} \gamma e^{-\gamma a_2} \\ &\geq (y''/8)^3 \frac{1}{8} (1/y'') e^{-\gamma y''} = (y'')^2 2^{-12} e^{-1} = 2^{-12} e^{-1} \frac{1}{\gamma^2}. \end{aligned} \quad (5.22)$$

Next suppose that $1/\gamma \geq y'$ and let $y'' = 1/\gamma$. Then for any $b_0 \in \mathbb{R}$ there exists an interval $(b_1, b_2) \subset (2y'', 6y'')$ such that $b_2 - b_1 \geq y''$ and for any $y \in (b_1, b_2)$, we have $|y - b_0| \geq y''$. We will apply this observation with $b_0 = \mathbb{E}W_2$. Since $y' \geq R$ and $y'' \geq y'$, we have $b_1 \geq 2y'' \geq y' + R$. Hence, we can apply (5.19) to $y \in (b_1, b_2)$ and obtain

$$\begin{aligned} \text{Var } W_2 &\geq \int_{b_1}^{b_2} |y - \mathbb{E}W_2|^2 f_{W_2}(y) dy \geq \int_{b_1}^{b_2} (y'')^2 \gamma e^{-\gamma y} dy \geq (b_2 - b_1)(y'')^2 \gamma e^{-\gamma b_2} \\ &\geq (y'')^3 (1/y'') e^{-6\gamma y''} = (y'')^2 e^{-6} = e^{-6} \frac{1}{\gamma^2}. \end{aligned}$$

Since $e^{-6} > 2^{-12}e^{-1}$, the above estimate, (5.17) and (5.22) imply that

$$w/q - u^2 = \text{Var } W_2 \geq 2^{-12}e^{-1} \frac{1}{\gamma^2}.$$

This yields the lower bound in (5.9).

The bound in (5.10) follows from (5.8) and (5.9). $\square$

Lemma 5.2. (i) For any $y \geq R$, the function $\lambda \rightarrow u(\lambda, y)$ is strictly decreasing.

(ii) For every $u > 0$ and $y \geq R$ there exists a unique $\lambda > 0$ satisfying $u = u(\lambda, y)$.

Proof. Recall $\hbar_n$ defined in (5.1) and formula (5.2). We have

$$\frac{\partial}{\partial \lambda} \left(\frac{\int_{D \setminus \mathcal{B}_*} r \lambda e^{-\lambda r} dx dr}{\int_{D \setminus \mathcal{B}_*} \lambda e^{-\lambda r} dx dr} \right) = \frac{\partial}{\partial \lambda} \left(\frac{\hbar_1}{\hbar_0} \right) = \frac{\hbar_1^2 - \hbar_2 \hbar_0}{\hbar_0^2},$$

which is strictly negative by the Cauchy-Schwarz inequality and, consequently, $\lambda \rightarrow u(\lambda, y) = \hbar_1(\lambda, y)/\hbar_0(\lambda, y)$ is strictly decreasing for every $y \geq R$. This proves (i).

Note that

$$\hbar_1(\lambda, y) = \int_{D \setminus \mathcal{B}_*} \lambda r e^{-\lambda r} dx dr \leq \int_D \lambda r e^{-\lambda r} dx dr = \frac{|D_b|}{\lambda}.$$

This and (5.7) imply that

$$u(\lambda, y) = \frac{\hbar_1(\lambda, y)}{\hbar_0(\lambda, y)} \leq \frac{|D_b|}{\lambda q} \leq \frac{8}{\lambda(1 - e^{1/2})} \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty.$$

On the other hand, using (5.7) again, when $\lambda \rightarrow 0$,

$$u(\lambda, y) = \frac{\hbar_1(\lambda, y)}{\hbar_0(\lambda, y)} = \frac{\hbar_1(\lambda, y)}{q(\lambda, y)} \geq \frac{|D_b|}{q} \int_{y+R}^{\infty} \lambda r e^{-\lambda r} dr \geq \frac{1}{\lambda} \int_{\lambda(y+R)}^{\infty} s e^{-s} ds \rightarrow \infty.$$

We conclude that for given $u > 0$ and $y \geq R$ there exists a unique $\lambda = \lambda(u, y) > 0$ such that $u = u(\lambda, y)$ holds. $\square$

Lemma 5.3. Recall that $\lambda(u, y)$ denotes the solution to $u = u(\lambda, y)$, and recall (5.3)–(5.6). We have

$$\frac{\partial \lambda}{\partial u} = \frac{q}{qu^2 - w}, \quad (5.23)$$

$$\frac{\partial \lambda}{\partial y} = \frac{\lambda u |D_b| - q}{qu^2 - w}, \quad (5.24)$$

$$\frac{\partial q}{\partial u} = \frac{q}{\lambda} (1 - \lambda u) \frac{\partial \lambda}{\partial u}, \quad (5.25)$$

$$\frac{\partial q}{\partial y} = \frac{q}{\lambda} (1 - \lambda u) \frac{\partial \lambda}{\partial y} + \lambda (|D_b| - q), \quad (5.26)$$

$$\frac{\partial}{\partial u} \left(\frac{qe^{\lambda u}}{\lambda} \right) = qe^{\lambda u}. \quad (5.27)$$

Proof. Comparing (5.1) and (5.3)–(5.6), we see that $q = h_0$, $uh_0 = h_1$ and $w = h_2$.

Using (5.2) and differentiating the identity

$$u\hbar_0(\lambda(u, y), y) = \hbar_1(\lambda(u, y), y) \quad (5.28)$$

with respect to u we obtain

$$\begin{aligned} \hbar_0 + u \left(\frac{\hbar_0}{\lambda} - \hbar_1 \right) \frac{\partial \lambda}{\partial u} &= \left(\frac{\hbar_1}{\lambda} - \hbar_2 \right) \frac{\partial \lambda}{\partial u}, \\ \frac{\partial \lambda}{\partial u} &= \hbar_0 \left(\frac{\hbar_1}{\lambda} - \hbar_2 - \frac{u\hbar_0}{\lambda} + u\hbar_1 \right)^{-1} = \frac{q}{qu^2 - w}, \end{aligned}$$

where we used the facts that $w = h_2$ and $uq = uh_0 = h_1$. This proves (5.23).

We prove (5.25) as follows,

$$\frac{\partial q}{\partial u} = \frac{\partial \hbar_0}{\partial \lambda} \frac{\partial \lambda}{\partial u} = \left(\frac{\hbar_0}{\lambda} - \hbar_1 \right) \frac{\partial \lambda}{\partial u} = \frac{q}{\lambda} (1 - \lambda u) \frac{\partial \lambda}{\partial u}.$$

We use the above formula in the following proof of (5.27),

$$\frac{\partial}{\partial u} \left(\frac{qe^{\lambda u}}{\lambda} \right) = \frac{e^{\lambda u}}{\lambda^2} \left(q(1 - \lambda u) \frac{\partial \lambda}{\partial u} + q\lambda \left(\lambda + u \frac{\partial \lambda}{\partial u} \right) - q \frac{\partial \lambda}{\partial u} \right) = qe^{\lambda u}.$$

Since $\int_0^\infty \lambda r^n e^{-\lambda r} dr = n!/\lambda^n$ and the volume of the $(d-1)$ -dimensional sphere with radius ρ is $\pi^{(d-1)/2} \rho^{d-1} / \Gamma(\frac{d+1}{2})$,

$$\begin{aligned} \hbar_n(\lambda, y) &= \frac{|D_b|n!}{\lambda^n} - \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} \int_{y-R}^{y+R} \lambda r^n e^{-\lambda r} (R^2 - (y-r)^2)^{(d-1)/2} dr \\ &= \frac{|D_b|n!}{\lambda^n} - \frac{\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})} e^{-\lambda y} \int_{-R}^R \lambda (s+y)^n e^{-\lambda s} (R^2 - s^2)^{(d-1)/2} ds. \end{aligned}$$

Consequently, by the dominated convergence theorem, we obtain

$$\frac{\partial \hbar_n}{\partial y} = \lambda \left(\frac{|D_b|n!}{\lambda^n} - \hbar_n \right) - n \left(\frac{|D_b|(n-1)!}{\lambda^{n-1}} - \hbar_{n-1} \right). \quad (5.29)$$

In particular,

$$\frac{\partial \hbar_0}{\partial y} = \lambda(|D_b| - \hbar_0), \quad \frac{\partial \hbar_1}{\partial y} = (|D_b| - \lambda \hbar_1) - (|D_b| - \hbar_0) = \hbar_0 - \lambda \hbar_1. \quad (5.30)$$

Differentiation of (5.28) with respect to y gives

$$u \left(\frac{\partial \hbar_0}{\partial \lambda} \frac{\partial \lambda}{\partial y} + \frac{\partial \hbar_0}{\partial y} \right) = \frac{\partial \hbar_1}{\partial \lambda} \frac{\partial \lambda}{\partial y} + \frac{\partial \hbar_1}{\partial y},$$

$$\frac{\partial \lambda}{\partial y} = \left(\frac{\partial \hbar_1}{\partial y} - u \frac{\partial \hbar_0}{\partial y} \right) \left(u \frac{\partial \hbar_0}{\partial \lambda} - \frac{\partial \hbar_1}{\partial \lambda} \right)^{-1}.$$

We now use (5.2) and (5.30) to see that

$$\frac{\partial \lambda}{\partial y} = \frac{\lambda u |D_b| - q}{qu^2 - w}.$$

This proves (5.24).

We apply (5.2) and (5.30) once again to prove (5.26),

$$\frac{\partial q}{\partial y} = \frac{\partial \hbar_0}{\partial \lambda} \frac{\partial \lambda}{\partial y} + \frac{\partial \hbar_0}{\partial y} = \frac{q}{\lambda} (1 - \lambda u) \frac{\partial \lambda}{\partial y} + \lambda (|D_b| - q).$$

□

Proof of Theorem 2.3. In view of (5.3)–(5.4), the Eqs. (2.8)–(2.9) can be written in this form,

$$M = K(\lambda, y) := m \frac{|D_b| - q(\lambda, y)}{q(\lambda, y)} = m \left(\frac{|D_b|}{q(\lambda, y)} - 1 \right), \quad (5.31)$$

$$G(\lambda, y) := \frac{dmg}{2\lambda} + mg u(\lambda, y) + Mgy = E. \quad (5.32)$$

It is easy to see that all partial derivatives of any order of the functions $K(\lambda, y)$ and $G(\lambda, y)$ exist and are continuous.

Consider any $y \geq R$ such that $Mgy < E$. For sufficiently small $\lambda > 0$, $dmg/(2\lambda) > E$, so for some $\lambda > 0$, $G(\lambda, R) > E$. Our assumption that $Mgy < E$ and (5.8) imply that for very large λ , $G(\lambda, R) < E$. By continuity of $G(\lambda, y)$, there exists λ such that $G(\lambda, y) = E$. Let λ_y denote the smallest λ with this property.

Part (i) of the lemma holds true because we have assumed that $MgR < E$. Therefore, we can take $\lambda_* = \lambda_R$.

We have $\lim_{y \uparrow E/(Mg)} \lambda_y = \infty$ because the term Mgy in the formula for $G(\lambda, y)$ approaches E , so the first term, $dmg/(2\lambda_y)$, must go to 0.

By the assumptions of part (ii) of the lemma, $K(\lambda_R, R) = K(\lambda_*, R) > M$. It is easy to see that the function $q(\lambda, y)$ converges to $|D_b|$ when $\lambda \rightarrow \infty$, no matter how y and λ are related. Hence $\lim_{y \uparrow E/(Mg)} K(\lambda_y, y) = 0$. By continuity of $K(\lambda, y)$, there exists y such that $K(\lambda_y, y) = M$. Let y_A be the smallest y with this property and let $\lambda_A = \lambda_{y_A}$. Note that $R < y_A < E/(Mg)$.

It remains to prove uniqueness of the solution (y, λ) to (2.8)–(2.9).

Let $z(y) = \frac{E - Mgy}{mg}$. By Lemma 5.2, the function

$$\kappa_y(u) := u + \frac{d}{2\lambda(u, y)} - z(y)$$

is a strictly increasing function of u . By (5.8), $\kappa_y(0^+) = -z(y)$ and $\kappa_y(z(y)) > 0$, so for every given $y \geq R$ there exists a unique $u = u(y)$ such that

$$u(y) = z(y) - \frac{d}{2\lambda(u(y), y)}. \quad (5.33)$$

Comparing this formula to (5.32), we see that we must have $u(\lambda, y) = u(y)$. It will suffice to show that there is at most one y such that $q(\lambda(u(y), y), y) = q(u(y), y)$ satisfies (5.31).

Assuming that $u(y)$ satisfies (5.33),

$$\frac{du}{dy} = -\frac{M}{m} + \frac{d}{2\lambda^2} \left(\frac{\partial \lambda}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \lambda}{\partial y} \right).$$

Hence,

$$\begin{aligned} \frac{du}{dy} \left(1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u} \right) &= -\frac{M}{m} + \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial y}, \\ \frac{du}{dy} &= \frac{-\frac{M}{m} + \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial y}}{1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u}}. \end{aligned} \quad (5.34)$$

We use this formula and (5.25)–(5.26) to see that

$$\begin{aligned} \frac{d}{dy} q(u(y), y) &= \frac{\partial q}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial q}{\partial y} = \frac{q}{\lambda} (1 - \lambda u) \left(\frac{\partial \lambda}{\partial u} \frac{du}{dy} + \frac{\partial \lambda}{\partial y} \right) + \lambda(|D_b| - q) \quad (5.35) \\ &= \frac{q}{\lambda} (1 - \lambda u) \left(\frac{\partial \lambda}{\partial u} \frac{-\frac{M}{m} + \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial y}}{1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u}} + \frac{\partial \lambda}{\partial y} \right) + \lambda(|D_b| - q). \end{aligned}$$

Lemma 5.2 (i) implies that $\partial \lambda / \partial u < 0$ so $1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u} > 0$. Since we are interested only in the sign of dq/dy , it will suffice to analyze $A := \frac{dq}{dy} \left(1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u} \right)$. Multiplying both sides of (5.35) by $1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u}$, we obtain

$$\begin{aligned} A &= \frac{q}{\lambda} (1 - \lambda u) \left(\frac{\partial \lambda}{\partial u} \left(-\frac{M}{m} + \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial y} \right) + \frac{\partial \lambda}{\partial y} \left(1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u} \right) \right) \\ &\quad + \lambda(|D_b| - q) \left(1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u} \right) \\ &= \frac{q}{\lambda} (1 - \lambda u) \left(-\frac{M}{m} \frac{\partial \lambda}{\partial u} + \frac{\partial \lambda}{\partial y} \right) + \lambda(|D_b| - q) \left(1 - \frac{d}{2\lambda^2} \frac{\partial \lambda}{\partial u} \right) \\ &= \frac{q}{\lambda(w - qu^2)} \left((1 - \lambda u) \left(\frac{Mq}{m} + q - \lambda u|D_b| \right) + \frac{d}{2} (|D_b| - q) \right) + \lambda(|D_b| - q) \\ &= \frac{q|D_b|}{\lambda(w - qu^2)} \\ &\quad \left((1 - \lambda u)^2 + (1 - \lambda u) \left(\frac{q}{|D_b|} (1 + M/m) - 1 \right) + \frac{d}{2} \left(1 - \frac{q}{|D_b|} \right) \right) + \lambda(|D_b| - q). \end{aligned}$$

If $(q/|D_b|)(1 + M/m) - 1 = 0$ then

$$A = \frac{q|D_b|}{\lambda(w - qu^2)} \left((1 - \lambda u)^2 + \frac{d}{2} \left(1 - \frac{q}{|D_b|} \right) \right) + \lambda(|D_b| - q).$$

According to (5.17), $w - qu^2 > 0$. Since we assume that $R > 0$, $|D_b| - q > 0$. It follows that $A > 0$ and, therefore, $dq/dy > 0$ if $(q/|D_b|)(1 + M/m) - 1 = 0$. In other words, $dq/dy > 0$ whenever $q(u(y), y) = |D_b|/(1 + M/m)$. A smooth function cannot cross a level multiple times if its derivative is strictly positive at every crossing point. We have proved that there is at most one y such that $q(u(y), y)$ satisfies (5.31). $\square$

Recall (2.1) and for $u > 0$, $y \geq R$ and $x \in D'_b$ let

$$\begin{aligned} \mathbf{D}_n(y, u) &= \left\{ (\mathbf{x}_n, x_{n+1}, \mathbf{y}_n) \in D_b^n \times D'_b \times \mathbb{R}^n : \frac{1}{n} \sum_{i=1}^n y_i = u, \right. \\ &\quad \left. (x_k, y_k) \in D \setminus \mathcal{B}((x_{n+1}, y), R), k = 1, \dots, n \right\}, \\ \mathbf{D}_n(x, y, u) &= \left\{ (\mathbf{x}_n, \mathbf{y}_n) \in D_b^n \times \mathbb{R}^n : \frac{1}{n} \sum_{i=1}^n y_i = u, \right. \\ &\quad \left. (x_k, y_k) \in D \setminus \mathcal{B}((x, y), R), k = 1, \dots, n \right\}. \end{aligned} \quad (5.36)$$

Proposition 5.4. Consider $u > 0$, $y \geq R$ and $x \in D'_b$. Let $\text{Vol}_n(x, y, u)$ be the $(nd - 1)$ -dimensional volume of $\mathbf{D}_n(x, y, u)$ and $\text{Vol}_n(y, u)$ be the $((n + 1)d - 2)$ -dimensional volume of $\mathbf{D}_n(y, u)$. Let $\lambda = \lambda(u, y) > 0$ be the solution to $u = u(\lambda, y)$ and recall $q = q(u, y)$ defined in (5.3) and (5.6). Then for some absolute constants $0 < c_1, c_2, c_3, c_4 < \infty$ and n_1 , for all $n \geq n_1$,

$$c_1 |D_b| n^{1-n} \left(\frac{qe^{\lambda u}}{\lambda} \right)^{n-1} \leq \text{Vol}_n(x, y, u) \leq c_2 |D_b| n^{1-n} \left(\frac{qe^{\lambda u}}{\lambda} \right)^{n-1}, \quad (5.37)$$

$$c_3 |D'_b| \cdot |D_b| n^{1-n} \left(\frac{qe^{\lambda u}}{\lambda} \right)^{n-1} \leq \text{Vol}_n(y, u) \leq c_4 |D'_b| \cdot |D_b| n^{1-n} \left(\frac{qe^{\lambda u}}{\lambda} \right)^{n-1}. \quad (5.38)$$

Proof. First note that (5.38) is an immediate corollary of (5.37) so it will suffice to prove (5.37).

Let $\mathcal{B}_* = \mathcal{B}((x, y), R)$,

$$\begin{aligned} H &= \{(z_1, t_1, \dots, z_n, t_n) : (t_1 + \dots + t_n)/n = u; t_k > 0, z_k \in D_b, \text{ for } 1 \leq k \leq n\}, \\ H_* &= H \cap (D \setminus \mathcal{B}_*)^n. \end{aligned}$$

Let $(Z_1^1, T_1^1), \dots, (Z_n^1, T_n^1)$ be i.i.d., with $Z_i^1 \in \mathbb{R}^{d-1}$ and $T_i^1 > 0$ for $1 \leq i \leq n$. Assume that T_i^1 and Z_i^1 are independent, Z_i^1 has the uniform distribution in D_b , and T_i^1 has the exponential distribution with parameter λ .

Let $f_1(z_1, t_1, \dots, z_n, t_n)$ be the density of $((Z_1^1, T_1^1), \dots, (Z_n^1, T_n^1))$. We have

$$f_1(z_1, t_1, \dots, z_n, t_n) = |D_b|^{-n} \lambda^n \exp(-\lambda(t_1 + \dots + t_n)), \quad (5.39)$$

for $z_i \in D_b, t_i > 0, 1 \leq i \leq n$. Note that the density f_1 is constant on H .

Let $(Z_1^2, T_1^2), \dots, (Z_n^2, T_n^2)$ be i.i.d., with (Z_i^2, T_i^2) being distributed as (Z_i^1, T_i^1) conditioned by $\{(Z_i^1, T_i^1) \notin \mathcal{B}_*\}$. By the definition of q and (5.39), the density f_2 of $((Z_1^2, T_1^2), \dots, (Z_n^2, T_n^2))$ is given by

$$f_2(z_1, t_1, \dots, z_n, t_n) = \left(\frac{q}{|D_b|} \right)^{-n} |D_b|^{-n} \lambda^n \exp(-\lambda(t_1 + \dots + t_n)) \quad (5.40)$$

on the set $(D \setminus \mathcal{B}_*)^n$.

Let $S_n^1 = \frac{1}{n} \sum_{k=1}^n T_k^1 = \sum_{k=1}^n (T_k^1/n)$. The distribution of T_k^1/n is exponential with mean $(n\lambda)^{-1}$, so the distribution of S_n^1 is gamma with the density $f_{S_n^1}(s) = ((n\lambda)^n / \Gamma(n)) s^{n-1} e^{-n\lambda s}$. Hence,

$$f_{S_n^1}(u) = \frac{(n\lambda)^n}{\Gamma(n)} u^{n-1} e^{-n\lambda u}. \quad (5.41)$$

Let $S_n^2 = \frac{1}{n} \sum_{k=1}^n T_k^2$ and $T_j^3 = T_j^2 - u$ for $1 \leq j \leq n$. It follows from the fact that $u = u(\lambda, y)$ that $\mathbb{E}T_j^2 = u$ and $\mathbb{E}T_j^3 = 0$. Let $\sigma^2 = \text{Var } T_j^2 = \text{Var } T_j^3$ and $S_n^3 = \frac{1}{n^{1/2}\sigma} \sum_{k=1}^n T_k^3 = n^{1/2}\sigma^{-1}(S_n^2 - u)$. We have

$$f_{S_n^2}(s) = \sigma^{-1} n^{1/2} f_{S_n^3}(\sigma^{-1} n^{1/2}(s - u)).$$

By Lemma 3.1,

$$f_{S_n^2}(u) = \sigma^{-1} n^{1/2} \left(\frac{1}{\sqrt{2\pi}} + A \right), \quad (5.42)$$

where

$$|A| \leq \frac{C \left(\mathbb{E}|T_j^3/\sigma|^3 \right)^2 \max(1, C_1^3)}{\sqrt{n}}, \quad (5.43)$$

$$C_1 = \sup_{u \in \mathbb{R}} f_{T_j^3/\sigma}(u). \quad (5.44)$$

In view of (5.39), (5.40), (5.41) and (5.42),

$$\frac{\text{Vol}(H_*)}{\text{Vol}(H)} = \frac{q^n f_{S_n^2}(u)}{|D_b|^n f_{S_n^1}(u)} = \frac{q^n \sigma^{-1} n^{1/2} \left(\frac{1}{\sqrt{2\pi}} + A \right)}{|D_b|^n \frac{(n\lambda)^n}{\Gamma(n)} u^{n-1} e^{-n\lambda u}}. \quad (5.45)$$

The volume of a regular n -simplex with unit side length is $\sqrt{n}/((n-1)!2^{(n-1)/2})$. The volume of a regular n -simplex with the side length $\sqrt{2}$ is $(\sqrt{2})^{n-1} \sqrt{n}/((n-1)!2^{(n-1)/2}) = \sqrt{n}/(n-1)!$. Hence $\text{Vol}(H) = |D_b|^n u^{n-1} \sqrt{n}/(n-1)!$. This and (5.45) imply that

$$\text{Vol}(H_*) = \text{Vol}(H) \frac{q^n \sigma^{-1} n^{1/2} \left(\frac{1}{\sqrt{2\pi}} + A \right)}{|D_b|^n \frac{(n\lambda)^n}{\Gamma(n)} u^{n-1} e^{-n\lambda u}} = \frac{|D_b|^n u^{n-1} \sqrt{n} q^n \sigma^{-1} n^{1/2} \left(\frac{1}{\sqrt{2\pi}} + A \right)}{(n-1)! |D_b|^n \frac{(n\lambda)^n}{\Gamma(n)} u^{n-1} e^{-n\lambda u}} \quad (5.46)$$

$$= \frac{q^n \sigma^{-1} n \left(\frac{1}{\sqrt{2\pi}} + A \right)}{(n\lambda)^n e^{-n\lambda u}} = \left(\frac{q e^{\lambda u}}{\lambda} \right)^n \left[n^{1-n} \sigma^{-1} \left(\frac{1}{\sqrt{2\pi}} + A \right) \right].$$

In this proof, for any functions a and b of any number of variables, we will write $a \approx b$ to indicate that there exist universal constants $0 < c', c'' < \infty$ such that $c' \leq a/b \leq c''$ for all values of the arguments.

By Lemma 5.1, $q \approx |D_b|$, $\lambda u \approx 1$ and $\lambda \sigma = \lambda \sqrt{w/q - u^2} \approx 1$, so (5.46) implies that

$$\text{Vol}(H_*) \approx \left(\frac{q e^{\lambda u}}{\lambda} \right)^{n-1} \left[|D_b| n^{1-n} \left(\frac{1}{\sqrt{2\pi}} + A \right) \right]. \quad (5.47)$$

Next we will find an upper bound for A defined in (5.42) using (5.43)–(5.44).

We use the bound $f_{T_j^2}(y) \leq \frac{|D_b|}{q} \lambda e^{-\lambda y}$ and the substitution $\lambda r = s$ to obtain

$$\mathbb{E}|T_j^3/\sigma|^3 \leq \sigma^{-3} \frac{|D_b|}{q} \int_0^\infty |r - u|^3 \lambda e^{-\lambda r} dr = \frac{1}{(\lambda \sigma)^3} \frac{|D_b|}{q} \int_0^\infty |s - u\lambda|^3 e^{-s} ds.$$

We have already pointed out that $\lambda \sigma \approx 1$, $|D_b|/q \approx 1$ and $u\lambda \approx 1$. The last formula shows that $\mathbb{E}|T_j^3/\sigma|^3 \leq c_5 < \infty$, where c_5 is a universal constant.

Recall that $\lambda \sigma \approx 1$ and $|D_b|/q \approx 1$ to see that for some universal constant $c_6 < \infty$,

$$f_{T_j^3/\sigma}(s) \leq q^{-1} \sigma \lambda e^{-\lambda(s\sigma+u)} \leq \sigma \lambda q^{-1} \leq \frac{c_6}{|D_b|}.$$

This bound and $\mathbb{E}|T_j^3/\sigma|^3 \leq c_5$ imply that $|A| \leq c_7 / (\sqrt{n} \min(1, |D_b|^3))$ where c_7 is an absolute constant.

Consider the case $|D_b| = 1$. Then $|A| \leq c_7/\sqrt{n}$. Let n_1 be so large that for $n \geq n_1$,

$$\frac{1}{2\sqrt{2\pi}} < \frac{1}{\sqrt{2\pi}} + \frac{c_7}{\sqrt{n}} < \frac{2}{\sqrt{2\pi}}.$$

This and (5.47) prove (5.37) in the case $|D_b| = 1$.

For $|D_b| \neq 1$, we use scaling. If x, y, D, R and u are multiplied by $c_* > 0$, it is elementary to verify that $\text{Vol}_n(x, y, u), q, \lambda$ and u are rescaled by powers of c_* such that (5.37) remains true. $\square$

Remark 5.5. We believe that Proposition 5.4 holds for all n . One could prove the bounds for $n < n_1$ using elementary estimates similar to those in the proof of Lemma 5.1. We do not provide a proof because it is not needed for our main theorem.

6. Archimedes' Principle

In this section, for any functions $a_1(\cdot)$ and $a_2(\cdot)$ of any arguments we will write

$$a_1(\cdot) \approx a_2(\cdot) \iff \exists 0 < c_1, c_2 < \infty : c_1 a_1(\cdot) \leq a_2(\cdot) \leq c_2 a_1(\cdot). \quad (6.1)$$

The constants c_1 and c_2 may depend only on “fixed” parameters in our model: d, D, D_b, R, M, m, g and E .

Proposition 6.1. *Recall the notation from Theorem 2.3 and assume that (2.10) holds. Recall (5.4) and let $u_A = u(\lambda_A, y_A)$. For every $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(|Y_{n+1} - y_A| > \varepsilon \text{ or } \left| \sum_{i=1}^n Y_i - u_A \right| > \varepsilon \right) = 0.$$

Proof. Recall notation from (2.6). The following formula for the marginal density $f_{Y_{n+1}}(y)$ of Y_{n+1} follows from (2.7),

$$f_{Y_{n+1}}(y) = \frac{1}{Z_n} \int_{\mathbf{D}_n^E(y)} \left(E - mg \frac{1}{n} \sum_{i=1}^n y_i - Mgy \right)^{((n+1)d-2)/2} d\mathbf{x}_{n+1} dy_n, \quad (6.2)$$

where Z_n is the normalizing constant. Let $z(y) = (E - Mgy)/(mg)$ and note that $z(y) > 0$ for $y \in [R, E/(Mg))$ because we have assumed that $E > MgR$. Recall notation from Proposition 5.4. The proposition implies that

$$f_{Y_{n+1}}(y) = \frac{1}{Z_n} \int_0^{z(y)} (E - Mgy - mgu)^{((n+1)d-2)/2} \text{Vol}_n(y, u) du \quad (6.3)$$

$$\approx \frac{1}{Z'_n} |D_b| n^{1-n} \int_0^{z(y)} (z(y) - u)^{((n+1)d-2)/2} \left(\frac{q(u, y) e^{\lambda(u, y)u}}{\lambda(u, y)} \right)^{n-1} du \quad (6.4)$$

where

$$Z'_n = |D_b| n^{1-n} \int_R^{E/(Mg)} dy \int_0^{z(y)} (z(y) - u)^{((n+1)d-2)/2} \left(\frac{q(u, y) e^{\lambda(u, y)u}}{\lambda(u, y)} \right)^{n-1} du.$$

The integral in (6.4) can be written as

$$\int_0^{z(y)} \alpha_y^{n-1}(u) (z(y) - u)^{d-1} du, \quad (6.5)$$

where $\alpha_y(u) = (z(y) - u)^{d/2} q e^{\lambda u} / \lambda$.

Let

$$\beta_y(u) = \lambda(u, y) - \frac{d}{2(z(y) - u)}. \quad (6.6)$$

By (5.27),

$$\frac{\partial}{\partial u} \alpha_y(u) = (z(y) - u)^{d/2} \frac{q e^{\lambda u}}{\lambda} \left(\lambda - \frac{d}{2(z(y) - u)} \right) = \alpha_y(y) \beta_y(u). \quad (6.7)$$

By Lemma 5.2, the function $u \rightarrow \lambda(u, y)$ is strictly decreasing. Hence $\beta_y(u)$ is strictly decreasing. When $u \downarrow 0$, $\lambda \rightarrow \infty$ by (5.8). Thus $\beta_y(0^+) = \infty$ for $y \in [R, E/(Mg))$. The estimate (5.8) implies that $\beta_y(z(y)^-) = -\infty$. We conclude that there is a unique $u_0 = u_0(y) \in (0, z(y))$ such that $\beta_y(u_0) = 0$, $\beta_y(y)$ is positive on $(0, u_0)$ and negative on $(u_0, z(y))$. In view of (6.7), $\alpha_y(u)$ attains its only maximum at u_0 on the interval $(0, u_0)$, and the maximum is strict. Note that

$$\beta_y(u) = \beta_y(u) - \beta_y(u_0) = \lambda(u, y) - \lambda(u_0, y) + \frac{d}{2} \frac{u_0 - u}{(z(y) - u_0)(z(y) - u)}. \quad (6.8)$$

This, and the facts that the function $u \rightarrow \lambda(u, y)$ is decreasing and $z(y) \geq z(y) - u > 0$ for $u \in (0, z(y))$, imply that

$$|\beta_y(u)| = -\beta_y(u) \geq \frac{d}{2z^2(y)} (u - u_0) = \frac{d}{2z^2(y)} |u - u_0|, \quad u \in (u_0, z(y)), \quad (6.9)$$

$$|\beta_y(u)| = \beta_y(u) \geq \frac{d}{2z^2(y)}(u_0 - u) = \frac{d}{2z^2(y)}|u - u_0|, \quad u \in (0, u_0). \quad (6.10)$$

To simplify notation, we will write $k = n - 1$. From (6.9)–(6.10), we obtain for $u \notin [u_0 - z(y)k^{-1/2}, u_0 + z(y)k^{-1/2}]$,

$$\frac{2\sqrt{k}z(y)|\beta_y(u)|}{d} = \frac{2}{d} \frac{z^2(y)|\beta_y(u)|}{z(y)k^{-1/2}} \geq \frac{2}{d} \frac{z^2(y)|\beta_y(u)|}{|u - u_0|} \geq 1.$$

This, (6.5) and (6.7), and the bound $z(y) - u < z(y)$ imply that

$$\begin{aligned} & \left(\int_0^{u_0 - z(y)k^{-1/2}} + \int_{u_0 + z(y)k^{-1/2}}^{z(y)} \right) \alpha_y^k(u)(z(y) - u)^{d-1} du \\ & \leq \left(\int_0^{u_0 - z(y)k^{-1/2}} + \int_{u_0 + z(y)k^{-1/2}}^{z(y)} \right) \alpha_y^k(u)(z(y) - u)^{d-1} \frac{2\sqrt{k}z(y)|\beta_y(u)|}{d} du \\ & = \frac{2z^d(y)\sqrt{k}}{d} \left(\int_0^{u_0 - z(y)k^{-1/2}} + \int_{u_0 + z(y)k^{-1/2}}^{z(y)} \right) \alpha_y^{k-1}(u)\alpha_y(u)\beta_y(u) du \\ & = \frac{2z^d(y)}{d\sqrt{k}} \left(\alpha_y^k\left(u_0 - \frac{z(y)}{\sqrt{k}}\right) + \alpha_y^k\left(u_0 + \frac{z(y)}{\sqrt{k}}\right) - \alpha_y^k(0) - \alpha_y^k(z(y)) \right) \\ & \leq \frac{4}{d\sqrt{k}} z^d(y) \alpha_y^k(u_0). \end{aligned} \quad (6.11)$$

We combine this estimate with the observation that $z(y) \leq E/mg$ and the following bound

$$\int_{u_0 - z(y)k^{-1/2}}^{u_0 + z(y)k^{-1/2}} \alpha_y^k(u)(z(y) - u)^{d-1} du \leq \frac{2z^d(y)}{\sqrt{k}} \alpha_y^k(u_0), \quad (6.12)$$

to arrive at

$$\int_0^{z(y)} \alpha_y^k(u)(z(y) - u)^{d-1} du \leq \left(2 + \frac{4}{d}\right) \frac{z^d(y)}{\sqrt{k}} \alpha_y^k(u_0). \quad (6.13)$$

Since $\beta_y(u) \leq 0$ for $u \in [u_0, z(y))$, (6.6) implies that, for $u \in [u_0, z(y))$,

$$u \geq \frac{z(y)}{1 + \frac{d}{2u\lambda(u,y)}}.$$

This and (5.8) yield,

$$u \geq \frac{z(y)}{1 + 100d}, \quad u \in [u_0, z(y)). \quad (6.14)$$

Since $\beta_y(u_0) = 0$, we obtain from (6.6),

$$u_0(y) = \frac{z(y)}{1 + \frac{d}{2u_0(y)\lambda(u_0(y),y)}},$$

and, by (5.8),

$$u_0(y) \leq \frac{z(y)}{1 + d/42} \leq \frac{21}{22} z(y). \quad (6.15)$$

Using (5.23), (5.8), (5.9) and (6.14), we get for $u \in [u_0, z(y))$,

$$\left| \frac{\partial \lambda}{\partial u} \right| = -\frac{\partial \lambda}{\partial u} = \frac{1}{(w/p - u^2)\lambda^2} (\lambda u)^2 \frac{1}{u^2} \leq 2^{12} e \cdot 21^2 \frac{(1 + 100d)^2}{z^2(y)} = \frac{c_1}{z^2(y)}, \quad (6.16)$$

where c_1 depends only on d .

For $u \in (u_0, u_0 + z(y)k^{-1/2})$ and $k \geq 44^2 = 1936$, in view of (6.15),

$$\begin{aligned} \frac{z^2(y)}{(z(y) - u_0)(z(y) - u)} &\leq \frac{z^2(y)}{(z(y) - \frac{21}{22}z(y))(z(y) - \frac{21}{22}z(y) - z(y)k^{-1/2})} \\ &\leq \frac{1}{(1/22)(1/44)} \leq 968. \end{aligned}$$

This estimate, (6.8) and (6.16) show that for $u \in (u_0, u_0 + z(y)k^{-1/2})$ and $k \geq 1936$,

$$\begin{aligned} -\beta_y(u) &= \lambda(u_0, y) - \lambda(u, y) + \frac{d}{2} \frac{u - u_0}{(z(y) - u_0)(z(y) - u)} \\ &\leq \frac{u - u_0}{z^2(y)} (c_1 + 484d) \leq \frac{c_2}{z(y)\sqrt{k}}, \end{aligned} \quad (6.17)$$

where $c_2 > 0$ depends only on d . For $u \in (u_0, u_0 + z(y)k^{-1/2})$ and $k \geq 1936$,

$$z(y) - u \geq z(y) - \frac{21}{22}z(y) - z(y)k^{-1/2} \geq z(y)/44.$$

This implies that, for $u \in (u_0, u_0 + z(y)k^{-1/2})$ and $k \geq 1936$,

$$\begin{aligned} &\int_0^{z(y)} \alpha_y^k(u) (z(y) - u)^{d-1} du \\ &\geq \alpha_y^k(u_0) \int_{u_0}^{u_0 + z(y)k^{-1/2}} (z(y) - u)^{d-1} \left(1 - \frac{\alpha_y(u_0) - \alpha_u(u)}{\alpha_y(u_0)} \right)^k du \\ &\geq z^{d-1}(y) \alpha_y^k(u_0) (1/44)^{d-1} \int_{u_0}^{u_0 + z(y)k^{-1/2}} \left(1 - \frac{\alpha_y(u_0) - \alpha_u(u)}{\alpha_y(u_0)} \right)^k du. \end{aligned} \quad (6.18)$$

Recall that u_0 is the maximum of α_y , and also (6.7) and (6.17). There exists $\tilde{u} \in (u_0, u_0 + z(y)k^{-1/2})$ such that,

$$\frac{\alpha_y(u_0) - \alpha_u(u)}{\alpha_y(u_0)} = -\frac{\alpha_y(\tilde{u})}{\alpha_y(u_0)} \beta_y(\tilde{u})(u - u_0) \leq \frac{c_2}{z(y)\sqrt{k}} (u - u_0) \leq \frac{c_2}{k}.$$

Hence, for $u \in (u_0, u_0 + z(y)k^{-1/2})$ and $k \geq 1936$,

$$\left(1 - \frac{\alpha_y(u_0) - \alpha_u(u)}{\alpha_y(u_0)} \right)^k \geq c_3.$$

We combine this with (6.18) to see that

$$\int_0^{z(y)} \alpha_y^k(y) (z(y) - u)^{d-1} du \geq \frac{c_4}{\sqrt{k}} z^d(y) \alpha_y^k(u_0). \quad (6.19)$$

Let $\lambda_0 = \lambda(u_0(y), y)$, $q_0 = q(u_0(y), y)$ and

$$\psi(y) = (2/d)^{d/2} \alpha_y(u_0) = (2/d)^{d/2} (z(y) - u_0(y))^{d/2} \frac{q_0 e^{\lambda_0 u_0}}{\lambda_0}. \quad (6.20)$$

Since $u_0(y)$ makes the right hand side of (6.6) equal to 0,

$$\psi(y) = (2/d)^{d/2} (z(y) - u_0(y))^{d/2} \frac{q_0 e^{\lambda_0 u_0}}{\lambda_0} = \frac{q_0 e^{\lambda_0 u_0}}{\lambda_0^{d/2+1}}. \quad (6.21)$$

Recall that $k = n - 1$. It follows from (6.4), (6.5), (6.13), (6.19) and (6.20) that

$$f_{Y_{n+1}}(y) \approx \frac{1}{Z''_n} \psi^{n-1}(y) z^d(y), \quad (6.22)$$

with the normalizing constant $Z''_n = \int_R^{E/(Mg)} \psi^{n-1}(y) z^d(y) dy$.

Comparing (5.33) and (6.6), we see that we can apply (5.34) to $u_0(y)$, i.e.,

$$\frac{du_0}{dy} \left(1 - \frac{d}{2\lambda_0^2} \frac{\partial \lambda}{\partial u} \right) = -\frac{M}{m} + \frac{d}{2\lambda_0^2} \frac{\partial \lambda}{\partial y}. \quad (6.23)$$

We use this formula, (5.25), (5.26) and (6.21) in the following calculation,

$$\begin{aligned} & \lambda_0^{d/2+2} e^{-\lambda_0 u_0} \frac{d\psi}{dy} \\ &= \lambda_0 \left(\frac{\partial q}{\partial u} \frac{du_0}{dy} + \frac{\partial q}{\partial y} \right) + \lambda_0^2 q_0 \frac{du_0}{dy} + q_0 \left(\lambda_0 u_0 - \frac{d+2}{2} \right) \left(\frac{\partial \lambda}{\partial u} \frac{du_0}{dy} + \frac{\partial \lambda}{\partial y} \right) \\ &= \frac{du_0}{dy} \left(\lambda_0 \frac{\partial q}{\partial u} + q_0 \left(\lambda_0 u_0 - \frac{d+2}{2} \right) \frac{\partial \lambda}{\partial u} + \lambda_0^2 q_0 \right) + \lambda_0 \frac{\partial q}{\partial y} + q_0 \left(\lambda_0 u_0 - \frac{d+2}{2} \right) \frac{\partial \lambda}{\partial y} \\ &= \frac{du_0}{dy} \left(\lambda_0 \frac{q_0}{\lambda_0} (1 - \lambda_0 u_0) \frac{\partial \lambda}{\partial u} + q_0 \left(\lambda_0 u_0 - \frac{d+2}{2} \right) \frac{\partial \lambda}{\partial u} + \lambda_0^2 q_0 \right) \\ &\quad + \lambda_0 \left(\frac{q_0}{\lambda_0} (1 - \lambda_0 u_0) \frac{\partial \lambda}{\partial y} + \lambda_0 (|D_b| - q_0) \right) + q_0 \left(\lambda_0 u_0 - \frac{d+2}{2} \right) \frac{\partial \lambda}{\partial y} \\ &= \lambda_0^2 q_0 \frac{du_0}{dy} \left(1 - \frac{d}{2\lambda_0^2} \frac{\partial \lambda}{\partial u} \right) + \lambda_0^2 (|D_b| - q_0) - \frac{dq_0}{2} \frac{\partial \lambda}{\partial y} \\ &= \lambda_0^2 q_0 \left(-\frac{M}{m} + \frac{d}{2\lambda_0^2} \frac{\partial \lambda}{\partial y} \right) + \lambda_0^2 (|D_b| - q_0) - \frac{dq_0}{2} \frac{\partial \lambda}{\partial y} \\ &= \frac{\lambda_0^2 q_0}{m} \left(m \frac{|D_b| - q_0}{q_0} - M \right). \end{aligned}$$

Recall the usual sign function that takes values $-1, 0$ or 1 . The formula given above implies that

$$\text{sign} \left(\frac{d\psi(y)}{dy} \right) = \text{sign} \left(m \frac{|D_b| - q_0(y)}{q_0(y)} - M \right) = \text{sign} \left(m \frac{\int_{\mathcal{B}((\tilde{x}, y), R)} \lambda_0 e^{-\lambda_0 r} dx dr}{\int_{D \setminus \mathcal{B}((\tilde{x}, y), R)} \lambda_0 e^{-\lambda_0 r} dx dr} - M \right). \quad (6.24)$$

According to (6.6) and the definition of $u_0(y)$,

$$\lambda(u_0(y), y) - \frac{d}{2(z(y) - u_0(y))} = 0. \quad (6.25)$$

This is equivalent to (2.9) (see also (5.32) and (5.33)). In view of (2.8)–(2.10), (6.24) and (6.25), by Theorem 2.3 (ii), there exists a unique $y_A \in (R, E/(Mg))$ such that $\left. \frac{d\psi(y)}{dy} \right|_{y=y_A} = 0$ and (6.25) holds. Hence $\psi(y)$ attains its maximum at y_A and it is strictly increasing on (R, y_A) and strictly decreasing on $(y_A, E/(Mg))$.

For later reference we note that the above argument also shows that $\lambda_0 = \lambda(u_0(y_A), y_A) = \lambda_A$, with λ_A as defined in Theorem 2.3, and, therefore, $u_0(y_A) = u_A$, with u_A as defined in the statement of the present theorem.

According to (6.22), for some c_5 and any $\varepsilon \in (0, \min(y_A - R, E/(Mg) - y_A))$,

$$\mathbb{P}_n(|Y_{n+1} - y_A| \geq \varepsilon) \leq \frac{c_5}{Z_n''} \left(\int_R^{y_A - \varepsilon} + \int_{y_A + \varepsilon}^{E/(Mg)} \right) \psi^n(y) z^{d/2}(y) dy.$$

Since

$$\begin{aligned} Z_n'' &\geq \int_{y_A}^{y_A + \varepsilon/2} \psi^n(y) z^{d/2}(y) dy + \int_{y_A - \varepsilon/2}^{y_A} \psi^n(y) z^{d/2}(y) dy \\ &\geq \frac{\varepsilon}{2} \psi^n(y_A + \varepsilon/2) z^{d/2}(y_A + \varepsilon/2) + \frac{\varepsilon}{2} \psi^n(y_A - \varepsilon/2) z^{d/2}(y_A), \end{aligned}$$

and

$$\begin{aligned} &\left(\int_R^{y_A - \varepsilon} + \int_{y_A + \varepsilon}^{E/(Mg)} \right) \psi^n(y) z^{d/2}(y) dy \leq (E/(Mg) - R) z^{d/2}(R) (\psi^n(y_A - \varepsilon) \\ &\quad + \psi^n(y_A + \varepsilon)), \end{aligned}$$

we have

$$\begin{aligned} \mathbb{P}_n(|Y_{n+1} - y_A| \geq \varepsilon) &\leq \frac{(E/(Mg) - R) z^{d/2}(R) (\psi^n(y_A - \varepsilon) + \psi^n(y_A + \varepsilon))}{\frac{\varepsilon}{2} \psi^n(y_A + \varepsilon/2) z^{d/2}(y_A + \varepsilon/2) + \frac{\varepsilon}{2} \psi^n(y_A - \varepsilon/2) z^{d/2}(y_A)} \\ &\leq \frac{2E}{\varepsilon Mg} \left(\frac{z(R)}{z(y_A + \varepsilon/2)} \right)^{d/2} \left[\left(\frac{\psi(y_A + \varepsilon)}{\psi(y_A + \varepsilon/2)} \right)^n + \left(\frac{\psi(y_A - \varepsilon)}{\psi(y_A - \varepsilon/2)} \right)^n \right]. \end{aligned}$$

The right-hand side goes to 0 as $n \rightarrow \infty$, i.e.,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n(|Y_{n+1} - y_A| \geq \varepsilon) = 0. \quad (6.26)$$

A calculation analogous to (6.11) gives for $\delta > 0$,

$$\begin{aligned} &\left(\int_0^{u_0 - \delta} + \int_{u_0 + \delta}^{z(y)} \right) \alpha_y^k(u) (z(y) - u)^{d-1} du \\ &\leq \frac{2z^d(y)}{d\sqrt{k}} \left(\alpha_y^k(u_0 - \delta) + \alpha_y^k(u_0 + \delta) - \alpha_y^k(0) - \alpha_y^k(z(y)) \right) \end{aligned} \quad (6.27)$$

$$\leq \frac{2z^d(y)}{d\sqrt{k}} \left(\alpha_y^k(u_0 - \delta) + \alpha_y^k(u_0 + \delta) \right).$$

Recall that $k = n - 1$ and, by (6.2) and (6.3), that $\frac{1}{n} \sum_{i=1}^n y_i = u$. The following remark made above about ψ applies also to α_y because of (6.20): “ $\psi(y)$ attains its maximum at y_A and it is strictly increasing on (R, y_A) and strictly decreasing on $(y_A, E/(Mg))$.” Combining (6.19) and (6.27), we obtain for every fixed y ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}_n \left(\left| \frac{1}{n} \sum_{i=1}^n Y_i - u_0(y) \right| > \delta \mid Y_{n+1} = y \right) \\ \leq \limsup_{k \rightarrow \infty} \frac{\frac{2z^d(y)}{d\sqrt{k}} \left(\alpha_y^k(u_0 - \delta) + \alpha_y^k(u_0 + \delta) \right)}{\frac{c_4}{\sqrt{k}} z^d(y) \alpha_y^k(u_0)} = 0. \end{aligned} \quad (6.28)$$

Using the fact that $u_0(y)$ is a continuous function of y (see, e.g., (6.23)), (6.26), (6.28) and applying the dominated convergence theorem to the indicator function of the event $\left\{ \left| \frac{1}{n} \sum_{i=1}^n Y_i - u_0(y) \right| > \varepsilon \right\}$ we obtain from (6.26) and (6.28),

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(|Y_{n+1} - y_A| > \varepsilon \text{ or } \left| \sum_{i=1}^n Y_i - u_0(y_A) \right| > \varepsilon \right) = 0.$$

It remains to recall that we have shown that $u_0(y_A) = u_A$. □

Recall definitions (2.2) of $(\mathbf{X}_n, \mathbf{Y}_n)$, (2.11) of $v_{x,y,\lambda}$, and (5.36) of $\mathbf{D}_n(x, y, u)$.

Lemma 6.2. Fix any $\hat{x} \in D'_b$, $u > 0$ and $\hat{y} \geq R$, and let $\hat{\mathcal{B}} = \mathcal{B}((\hat{x}, \hat{y}), R)$. Let $\mathbb{P}_n^{\hat{x}, \hat{y}, u}$ denote the uniform distribution on $\mathbf{D}_n(\hat{x}, \hat{y}, u)$. Suppose that $(\mathbf{X}_n, \mathbf{Y}_n)$ has the distribution $\mathbb{P}_n^{\hat{x}, \hat{y}, u}$.

- (i) When $n \rightarrow \infty$ then for every fixed $j \geq 1$, the distribution of (X_j, Y_j) converges to $v_{\hat{x}, \hat{y}, \lambda}$, where λ is the solution to $u = u(\lambda, y)$. Moreover, for any $j_1 \neq j_2$, the joint distribution of $((X_{j_1}, Y_{j_1}), (X_{j_2}, Y_{j_2}))$ converges to $v_{\hat{x}, \hat{y}, \lambda} \times v_{\hat{x}, \hat{y}, \lambda}$.
- (ii) The convergence in (i) is uniform in the sense that for any bounded d -dimensional rectangular parallelepipeds $A_1, A_2 \subset \mathbb{R}^d$ with non-empty interior; any $0 < u_1 < u_2 < \infty$, and any $\varepsilon > 0$, there exists n_1 such that for all $n \geq n_1$, $\hat{x} \in D'_b$, $y > R$ and $u \in [u_1, u_2]$,

$$\left| \mathbb{P}_n^{\hat{x}, y, u}(((X_{j_1}, Y_{j_1}), (X_{j_2}, Y_{j_2})) \in A_1 \times A_2) - v_{\hat{x}, y, \lambda} \times v_{\hat{x}, y, \lambda}(A_1 \times A_2) \right| < \varepsilon. \quad (6.29)$$

Proof. We will reuse some ideas from the proof of Proposition 5.4.

Let $(Z_1^1, T_1^1), \dots, (Z_n^1, T_n^1)$ be i.i.d., with $Z_i^1 \in \mathbb{R}^{d-1}$ and $T_i^1 > 0$ for $1 \leq i \leq n$. Assume that T_i^1 and Z_i^1 are independent, Z_i^1 has the uniform distribution in D_b , and T_i^1 has the exponential distribution with parameter λ .

Let $f_1(z_1, t_1, \dots, z_n, t_n)$ be the density of $((Z_1^1, T_1^1), \dots, (Z_n^1, T_n^1))$. We have

$$f_1(z_1, t_1, \dots, z_n, t_n) = |D_b|^{-n} \lambda^n \exp(-\lambda(t_1 + \dots + t_n)), \quad (6.30)$$

for $z_i \in D_b$, $t_i > 0$, $1 \leq i \leq n$.

Let $(Z_1^2, T_1^2), \dots, (Z_n^2, T_n^2)$ be i.i.d., with (Z_i^2, T_i^2) being distributed as (Z_i^1, T_i^1) conditioned by $\{(Z_i^1, T_i^1) \notin \widehat{\mathcal{B}}\}$. Hence, (Z_i^2, T_i^2) has the distribution $\nu_{\widehat{x}, \widehat{y}, \lambda}$.

By the definition of q and (6.30), the density f_2 of $((Z_1^2, T_1^2), \dots, (Z_n^2, T_n^2))$ is given by

$$f_2(z_1, t_1, \dots, z_n, t_n) = \left(\frac{q}{|D_b|} \right)^{-n} |D_b|^{-n} \lambda^n \exp(-\lambda(t_1 + \dots + t_n)) \quad (6.31)$$

on the set $(D \setminus \widehat{\mathcal{B}})^n$.

Let $S_n^2 = \frac{1}{n} \sum_{k=1}^n T_k^2$. Let $((Z_1^3, T_1^3), \dots, (Z_n^3, T_n^3))$ be the sequence $((Z_1^2, T_1^2), \dots, (Z_n^2, T_n^2))$ conditioned by $\{S_n^2 = u\}$. Note that the distribution of $((Z_1^3, T_1^3), \dots, (Z_n^3, T_n^3))$ is the uniform distribution on $\mathbf{D}_n(\widehat{x}, \widehat{y}, u)$. Hence, the distribution of (X_j, Y_j) under $\mathbb{P}_n^{\widehat{x}, \widehat{y}, u}$ is the same as the distribution of (Z_j^3, T_j^3) . We will show that the distribution of (Z_j^3, T_j^3) converges to that of (Z_j^2, T_j^2) as $n \rightarrow \infty$. This is equivalent to the weak convergence of $\mathbb{P}_n^{\widehat{x}, \widehat{y}, u}$ to $\nu_{\widehat{x}, \widehat{y}, \lambda}$ since the distribution of (Z_i^2, T_i^2) has the distribution $\nu_{\widehat{x}, \widehat{y}, \lambda}$.

Fix $j \geq 1$ and consider $n > j$. Let $S_n^{2,j} = \frac{1}{n-1} \sum_{k=1, \dots, n; k \neq j} T_k^2$.

Since $(Z_1^2, T_1^2), \dots, (Z_n^2, T_n^2)$ is an i.i.d. sequence, the density $f_{(Z_j^3, T_j^3)}$ has the form

$$f_{(Z_j^3, T_j^3)}(z, t) = c_1 f_{(Z_j^2, T_j^2)}(z, t) f_{S_n^{2,j}} \left(u + \frac{1}{n-1}(u-t) \right), \quad (6.32)$$

where c_1 is the normalizing constant.

Let $\sigma^2 = \text{Var } T_i^2$ and $T_i^4 = (T_i^2 - u)/(\sigma\sqrt{n-1})$. Note that, since $u = u(\lambda, y)$, $\mathbb{E}T_i^4 = 0$. Let

$$S_n^{4,j} = \sum_{k=1, \dots, n; k \neq j} T_k^4 = \frac{1}{\sigma\sqrt{n-1}} \sum_{k=1, \dots, n; k \neq j} (T_k^2 - u) = \frac{\sqrt{n-1}}{\sigma} (S_n^{2,j} - u).$$

Hence

$$f_{S_n^{2,j}}(u+s) = \frac{\sqrt{n-1}}{\sigma} f_{S_n^{4,j}} \left(\frac{\sqrt{n-1}}{\sigma} s \right).$$

By Lemma 3.1,

$$\begin{aligned} f_{S_n^{2,j}} \left(u + \frac{1}{n-1}(u-t) \right) &= \frac{\sqrt{n-1}}{\sigma} f_{S_n^{4,j}} \left(\frac{\sqrt{n-1}}{\sigma} \frac{1}{n-1}(u-t) \right) \\ &= \frac{\sqrt{n-1}}{\sigma} f_{S_n^{4,j}} \left(\frac{1}{\sigma\sqrt{n-1}}(u-t) \right) = \frac{\sqrt{n-1}}{\sigma} \left(\phi \left(\frac{1}{\sigma\sqrt{n-1}}(u-t) \right) + A \right). \end{aligned} \quad (6.33)$$

Since the random variable T_j^2 has the same distribution as the random variable with the same name in the proof of Proposition 5.4, the estimates (5.43)–(5.44) for A , and those at the end of the proof of Proposition 5.4 apply in the present case. Thus

$$|A| \leq \frac{c_2}{\sqrt{n-1}}, \quad (6.34)$$

where c_2 depends only on $|D_b|$.

Fix any $u_1 > 0$. Then, by (5.10) and (5.17), there exists $c_3 > 0$ such that for all $u \geq u_1$,

$$\sigma^2 \geq c_3. \quad (6.35)$$

Fix any $u_2 \in (u_1, \infty)$. It follows from (6.33), (6.34) and (6.35) that for any fixed $0 \leq t_1 < t_2 < \infty$, $u \in [u_1, u_2]$ and $\varepsilon > 0$ there exists n_1 such that for $n \geq n_1$, $t_3, t_4 \in [t_1, t_2]$, and $t_5 \notin [t_1, t_2]$,

$$1 - \varepsilon < \frac{f_{S_n^{2,j}}\left(u + \frac{1}{n-1}(u - t_3)\right)}{f_{S_n^{2,j}}(u)} < 1 + \varepsilon, \quad (6.36)$$

$$\frac{f_{S_n^{2,j}}\left(u + \frac{1}{n-1}(u - t_5)\right)}{f_{S_n^{2,j}}(u)} < 2. \quad (6.37)$$

This, (5.18) and (6.32) imply that the distribution of (Z_j^3, T_j^3) converges to that of (Z_j^2, T_j^2) as $n \rightarrow \infty$. Moreover, for any bounded d -dimensional rectangular parallelepiped $A_1 \subset \mathbb{R}^d$ with non-empty interior, any $0 < u_1 < u_2 < \infty$, and any $\varepsilon > 0$, there exists n_1 such that for all $n \geq n_1$, $\widehat{x} \in D'_b$, $y > R$ and $u \in [u_1, u_2]$,

$$\left| \mathbb{P}_n^{\widehat{x}, y, u}((X_{j_1}, Y_{j_1}) \in A_1) - \nu_{\widehat{x}, y, \lambda}(A_1) \right| < \varepsilon.$$

A completely analogous argument shows that for any $j_1 \neq j_2$, the distribution of $((Z_{j_1}^3, T_{j_1}^3), (Z_{j_2}^3, T_{j_2}^3))$ converges to that of $((Z_{j_1}^2, T_{j_1}^2), (Z_{j_2}^2, T_{j_2}^2))$ as $n \rightarrow \infty$, and also part (ii) of the lemma holds true. $\square$

Recall $\nu_{\widehat{x}, y, \lambda}$ defined in (2.11), the empirical measure $\mathbb{Q}_n$ defined in (2.12), and $\mathbf{D}_n(x, y, u)$ defined in (5.36).

Lemma 6.3. *The marginal distribution of X_{n+1} under $\mathbb{P}_n$ is uniform in D'_b . Given $\{X_{n+1} = x\}$, the conditional distribution of $\mathbb{Q}_n$ converges to ν_{x, y_A, λ_A} weakly, in probability as $n \rightarrow \infty$.*

Proof. It follows easily from the microcanonical ensemble formula (2.7) that the marginal distribution of X_{n+1} under $\mathbb{P}_n$ is uniform in D'_b .

The same formula (2.7) implies that one can represent $\mathbb{P}_n$ as follows. Let $\mathbb{P}_n^{x, y, u}$ denote the uniform distribution on $\mathbf{D}_n(x, y, u)$. Then there exists a probability measure μ_n on $D'_b \times [R, \infty) \times \mathbb{R}_+$ such that $\mathbb{P}_n = \int \mathbb{P}_n^{x, y, u} d\mu_n(x, y, u)$. Let $\mathbb{P}_n^x = \int \mathbb{P}_n^{x, y, u} \mu_n(x, dy, du)$. Let $\mathbb{E}_n^x$ and $\mathbb{E}_n^{x, y, u}$ denote expectations corresponding to $\mathbb{P}_n^x$ and $\mathbb{P}_n^{x, y, u}$.

Fix any $x \in D'_b$ and write $\nu'_x = \nu_{x, y_A, \lambda_A}$.

Fix any bounded d -dimensional rectangular parallelepiped $A \subset \mathbb{R}^d$ with non-empty interior. We have

$$\mathbb{E}_n^x (\mathbb{Q}_n(A) - \nu'_x(A))^2 = \mathbb{E}_n^x \left(\frac{1}{n} \sum_{i=1}^n \mathbf{1}_A(X_i, Y_i) - \nu'_x(A) \right)^2 \quad (6.38)$$

$$\begin{aligned}
&= \mathbb{E}_n^x \left(\frac{1}{n} \sum_{i=1}^n (\mathbf{1}_A(X_i, Y_i) - v'_x(A)) \right)^2 \\
&= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E}_n^x (\mathbf{1}_A(X_i, Y_i) - v'_x(A))^2 \\
&\quad + \frac{2}{n^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \mathbb{E}_n^x [(\mathbf{1}_A(X_i, Y_i) - v'_x(A)) (\mathbf{1}_A(X_j, Y_j) - v'_x(A))] \\
&= \frac{1}{n} \mathbb{E}_n^x (\mathbf{1}_A(X_1, Y_1) - v'_x(A))^2 \\
&\quad + \frac{2(n^2 - n)}{n^2} \mathbb{E}_n^x [(\mathbf{1}_A(X_1, Y_1) - v'_x(A)) (\mathbf{1}_A(X_2, Y_2) - v'_x(A))] \\
&= \frac{1}{n} \mathbb{E}_n^x (\mathbf{1}_A(X_1, Y_1) - v'_x(A))^2 \\
&\quad + \frac{2(n^2 - n)}{n^2} \left(\mathbb{E}_n^x [\mathbf{1}_A(X_1, Y_1) \mathbf{1}_A(X_2, Y_2)] - \mathbb{E}_n^x [\mathbf{1}_A(X_1, Y_1) v'_x(A)] \right. \\
&\quad \left. - \mathbb{E}_n^x [v'_x(A) \mathbf{1}_A(X_2, Y_2)] + (v'_x(A))^2 \right).
\end{aligned}$$

It is easy to see that the function $(x, y, u) \rightarrow \mathbb{P}_n^{x,y,u}$ is continuous in the weak topology. This, Proposition 6.1 and (6.29) imply that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \mathbb{E}_n^x [\mathbf{1}_A(X_1, Y_1) \mathbf{1}_A(X_2, Y_2)] \\
&= \lim_{n \rightarrow \infty} \int \mathbb{E}_n^{x,y,u} [\mathbf{1}_A(X_1, Y_1) \mathbf{1}_A(X_2, Y_2)] \mu_n(x, dy, du) = (v'_x(A))^2. \quad (6.39)
\end{aligned}$$

For the same reason,

$$\lim_{n \rightarrow \infty} \mathbb{E}_n^x [\mathbf{1}_A(X_k, Y_k)] = v'_x(A), \quad k = 1, 2. \quad (6.40)$$

The bound $\mathbb{E}_n^x (\mathbf{1}_A(X_1, Y_1) - v'_x(A))^2 \leq 1$, (6.38), (6.39) and (6.40) imply that

$$\lim_{n \rightarrow \infty} \mathbb{E}_n^x (\mathbb{Q}_n(A) - v'_x(A))^2 = 0.$$

The lemma follows because this statement holds for every A . $\square$

Proof of Theorem 2.4. The theorem follows from Proposition 6.1 and Lemma 6.3. $\square$

7. Inverse temperature

Proposition 7.1. *The inverse temperature of the gas is asymptotically proportional to λ_A defined in Theorem (2.3). More precisely, for every fixed $j \geq 1$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_n \left(\frac{1}{2} m \|V_j\|^2 \right) = \lim_{n \rightarrow \infty} \mathbb{E}_n \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{2} m \|V_i\|^2 \right) = \frac{dmg}{2\lambda_A}. \quad (7.1)$$

Proof. Since the total energy E is fixed, we have

$$\frac{1}{n} \sum_{i=1}^n mgY_i \leq E, \quad MgY_{n+1} \leq E, \quad \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{2} m \|V_i\|^2 \leq E.$$

Hence, these sequences of random variables are tight. Recall $\mathbb{P}_n^{x,y,u}$ and μ_n from the proof of Lemma 6.3. By Proposition 6.1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}_n \left(\frac{1}{n} \sum_{i=1}^n mgY_i \right) &= \lim_{n \rightarrow \infty} \int \mathbb{E}_n^{x,y,u} \left(\frac{1}{n} \sum_{i=1}^n mgY_i \right) d\mu_n(x, y, u) \\ &= \lim_{n \rightarrow \infty} \int u d\mu_n(x, y, u) = mg u_A = mg \frac{\int_{D \setminus \mathcal{B}((\tilde{x}, y_A), R)} y \lambda_A e^{-\lambda_A y} dx dy}{\int_{D \setminus \mathcal{B}((\tilde{x}, y_A), R)} \lambda_A e^{-\lambda_A y} dx dy}. \end{aligned} \quad (7.2)$$

Proposition 6.1 and tightness also imply that

$$\lim_{n \rightarrow \infty} \mathbb{E}_n MgY_{n+1} = Mgy_A. \quad (7.3)$$

This and (7.2) show that the expected value of the potential energy of the point particles and the ball converges to a fixed number. Since the total energy E is fixed, the expectation of the kinetic energy converges weakly to a fixed number as well, i.e., for some $\sigma > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E}_n \left(\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{2} m \|V_i\|^2 \right) = \sigma^2. \quad (7.4)$$

The total energy is fixed so (7.2), (7.3) and (7.4) show that

$$\sigma^2 + mg \frac{\int_{D \setminus \mathcal{B}((\tilde{x}, y_A), R)} y \lambda_A e^{-\lambda_A y} dx dy}{\int_{D \setminus \mathcal{B}((\tilde{x}, y_A), R)} \lambda_A e^{-\lambda_A y} dx dy} + Mgy_A = E.$$

Since y_A and λ_A solve (2.9), we must have $\sigma^2 = \frac{dmg}{2\lambda_A}$.

It remains to note that due to the symmetry of $\mu_{y_{n+1}}$ in (2.7),

$$\lim_{n \rightarrow \infty} \mathbb{E}_n \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{2} m \|V_i\|^2 \right) = \lim_{n \rightarrow \infty} \mathbb{E}_n \left(\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{2} m \|V_i\|^2 \right) = \sigma^2 = \frac{dmg}{2\lambda_A}.$$

□

8. Uniqueness of the Stationary Distribution

Proof of Theorem 2.1 (ii). The idea of the proof is the following. If there were more than one invariant measure, at least two of them would be mutually singular by Birkhoff's ergodic theorem ([Sin94]). Given any two starting configurations we will exhibit two deterministic trajectories meeting at the same point in the phase space at some time $t_1 > 0$. Then we will argue that due to the random nature of some reflections, both processes have densities that are strictly positive in some neighborhood of that point in the phase space. Hence, there are no mutually singular invariant measures.

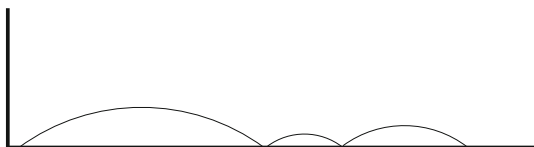


Fig. 1. Side view of a point particle trajectory reflecting from the bottom of the container

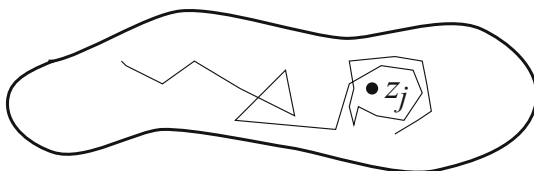


Fig. 2. View from above of a point particle trajectory reflecting from the bottom of the container

Much of the proof will be presented in a very informal way. This is because our argument is totally elementary but it would be extremely tedious to write (or read) in a fully rigorous way.

Step 1. Assume that the initial condition of the system does not belong to any of the families (1) and (2) described in the statement of the theorem. We will construct a single trajectory of the system. The trajectory will respect the laws of elastic collisions when they are assumed, i.e., for all collisions of the ball with the point particles and the walls of the container. For any reflection of a point particle from a wall of the container, we will choose a direction after the reflection from all possible directions in a way that meets the needs of the argument.

Recall that “walls” of the container include its bottom so point particles reflect according to the Lambertian law from the side walls and the bottom of the container.

Fix distinct points $z_1, \dots, z_{n+1}$ in D_b , such that the distance of z_{n+1} from the side wall of the container is greater than R .

We let the system evolve according to the original dynamics until one of the point particles hits a wall at a time s_1 . We assume that the particle that hit the wall is labeled 1, since the labeling of point particles is irrelevant.

The first point particle can reflect in any direction, including directions arbitrarily close to the boundary. So it can stay arbitrarily close to the boundary for an arbitrarily long time and move towards any point on the boundary, with the only limitation being its constant energy (the sum of potential and kinetic energies), see Figs. 1 and 2. We let the system evolve according to the original dynamics after time s_1 , except that the first point particle will follow its own trajectory, constructed independently. Let s_2 be the next time when a particle different from the first one hits a wall. We choose a trajectory for the first particle very close to the wall and moving towards z_1 in such a way that it avoids a collision with the ball on $[s_1, s_2]$ (recall that point particles do not interact).

We proceed by induction. Suppose that, for some $j < n$, a deterministic trajectory of the system has been constructed on $[0, s_j]$, including the trajectories of point particles labeled $1, \dots, j$. These point particles stay close to the walls from the time of the first hit of a wall until s_j . Particle k moves towards z_k from the first time it hits a wall until s_j , for $k = 1, \dots, j$. Given this inductive assumption, we let the system evolve according to the original dynamics after time s_j , except that point particles labeled $1, \dots, j$ will follow their own trajectories, constructed independently. Let s_{j+1} be the next time when

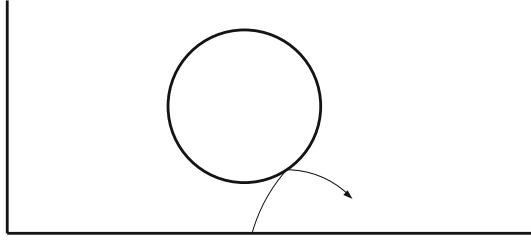


Fig. 3. Side view of the container. A point particle collides with the ball

a particle different from $1, \dots, j$ hits a wall. We will call this particle $j + 1$. We choose a trajectory for each of the particles $1, \dots, j + 1$ very close to the wall. The k -th particle is moving towards z_k in such a way that it avoids a collision with the ball on $[s_j, s_{j+1}]$, for $k = 1, \dots, j + 1$.

We stop the construction when we have a trajectory of the system on $[0, s_n]$. We will continue after discussing a delicate point in the next step.

Step 2. It is possible that fewer than n point particles hit the walls of the container. This can happen only if some particles always reflect from the top part of the ball. In this case, we let one of the point particles that are staying close to the boundary move towards the point where the ball reflects from the bottom of the container and then we let the point particle hit the ball slightly off center. That will nudge the ball off its trajectory. The result will be that the point particles formerly reflecting from the top of the ball will move to the side and eventually hit a wall.

Step 3. At this step of the construction of the trajectory of the system, point particles will come close to the point at the bottom where the ball reflects and they will hit the ball, one at a time, see Fig. 3. All other point particles will keep close to the bottom and stay away from the ball.

There are two major goals of the construction that can be achieved by this procedure. First, we can achieve equipartition of the energy. Second, we can put the ball above z_{n+1} and make it move vertically.

We will explain how we can arrange for equipartition of energy between the point particles and the ball, so that $(E - MgR)/(n + 1)$ of energy is given to each point particle and $(E - MgR)/(n + 1) + MgR$ of energy is given to the ball. Note that the minimal amount of energy a point particle can have is 0, assuming that it is sitting motionless at the bottom. For the ball, the minimal amount of energy is MgR .

The point particles do not interact with each other so the only way to transfer the energy between them is via collisions with the macroscopic ball. We let point particles approach the point where the ball reflects from the bottom, one at a time. Then we make the direction of the velocity of the point particle close to vertical (or at least considerably different from the horizontal; see Fig. 3). We make the point particle hit the ball either when the ball is moving up or down. In the first case, the point particle will lose energy and in the second case it will acquire energy. By manipulating the place of the collision and the velocity direction of the point particle before the collision, and by repeating the procedure, if necessary, many times, we can partition the energy between particles and the ball in an arbitrary way.

We need to add a few words clarifying the algorithm described above. If a particle and the ball have the same amount of kinetic energy and n is large then the speed of the particle is much larger than the speed of the ball. Hence, we can start by transferring

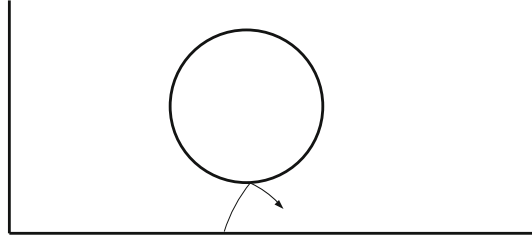


Fig. 4. Side view of the container. A point particle hits the ball at the lowest point on the surface. The scattering rule for the collision of the particle with the ball is totally elastic (the total energy and total momentum of the two-body system are conserved)

energy to the ball from all point particles that have more than $(E - MgR)/(n + 1)$ of energy. Then the energy can be transferred from the ball to the particles that had less than $(E - MgR)/(n + 1)$ of energy, one by one.

Step 4. We make the n -th point particle collide with the ball as depicted in Fig. 3 to change the trajectory of the ball so it reflects vertically at $z_{n+1} \in D_b$.

After this is done, energy might not be equidistributed. If necessary, we induce energy transfer between the n -th particle and the ball by collisions of the particle with the bottom of the ball, as in Fig. 4.

Fix some time t_1 greater than the duration of the trajectory constructed so far, such that the ball hits z_{n+1} at time t_1 . Make all point particles follow trajectories such that they all hit their own base points z_k , $k = 1, \dots, n$, at time t_1 . For this to be possible, it may be necessary to move t_1 to one of the later times when the ball hits z_{n+1} .

For future reference, let the deterministic trajectory constructed above be called $\Gamma = \{\Gamma(t), 0 \leq t \leq t_1\}$.

Step 5. To finish the proof, we will argue as follows. We have shown that the system can get to the same configuration at time t_1 for every initial configuration (the time t_1 may depend on the initial conditions—this is not a problem). The trajectory that we constructed is deterministic but it “agrees” with the dynamics of the system, including Lambertian reflections. Lambertian reflections introduce randomness. They make the state of the system at time t_1 random, with a density. The densities for different initial configurations overlap so there is only one stationary measure.

The subtle point is that the state density at time t_1 is not with respect to Lebesgue measure on $\mathbb{R}^{2nd}$ but on a hypersurface of dimension $2nd - 1$ because the total energy is fixed (see (2.3)). We will now outline an argument addressing this concern.

Recall notation from Sect. 2 and let

$$\begin{aligned} \mathbf{Z}_j(t) &= ((X_1(t), \dots, X_j(t)), (Y_1(t), \dots, Y_j(t)), (V_1(t), \dots, V_j(t))), \\ \mathbf{Z}_{j+}(t) &= ((X_j(t), \dots, X_{n+1}(t)), (Y_j(t), \dots, Y_{n+1}(t)), (V_j(t), \dots, V_{n+1}(t))), \\ \mathbf{z}_j &= ((x_1, \dots, x_j), (y_1, \dots, y_j), (v_1, \dots, v_j)), \\ \mathbf{z}_{j+} &= ((x_j, \dots, x_{n+1}), (y_j, \dots, y_{n+1}), (v_j, \dots, v_{n+1})), \end{aligned}$$

where $x_k \in \mathbb{R}^{d-1}$, $y_k \in R$ and $v_k \in \mathbb{R}^d$.

For arbitrarily thin tube around Γ there is a strictly positive probability that the system with random reflections $\mathbf{Z}_{n+1}$ will stay inside the tube until time t_1 . Consider a tube so thin that point particles undergoing random reflections in the tube collide with the ball in the same order as along the deterministic trajectory Γ .

We will assume without loss of generality that the point particles exchange energy with the ball along Γ in the order $1, \dots, n$.

Let u_1 be the last time the first particle hits a wall of the container before starting the process of exchanging the energy with the ball. If the tube is very thin, the first particle will not collide with the ball after exchanging the energy with the ball. Let u'_1 be the last time the first ball collides with the ball before time t_1 . We claim that for some small neighborhood U_1 of $\Gamma(t_1)$, some $c_1 > 0$ and $\mathbf{z}_{n+1} = (\mathbf{z}_1, \mathbf{z}_{2+}) \in U_1$,

$$\mathbb{P}_n(\mathbf{Z}_1(t_1) \in d\mathbf{z}_1 \mid \mathbf{Z}_{n+1}(u_1)) / d\mathbf{z}_1 \geq c_1, \quad (8.1)$$

$$\mathbb{P}_n(\mathbf{Z}_{n+1}(t_1) \in d\mathbf{z}_{n+1} \mid \mathbf{Z}_{n+1}(u'_1)) / d\mathbf{z}_{n+1} \geq c_1 \mathbb{P}_n(\mathbf{Z}_{2+}(t_1) \in d\mathbf{z}_{2+} \mid \mathbf{Z}_{n+1}(u'_1)) / d\mathbf{z}_{2+}. \quad (8.2)$$

The claim (8.1) is true because the first particle acquires a random amount of energy, and its position and velocity direction are random due to Lambertian reflections following u_1 . The claim (8.2) is a form of independence for conditional processes (conditioned to stay in separate tubes).

We proceed by induction. Suppose $2 \leq j \leq n-1$. Let u_j be the last time the j -th particle hit a wall of the container before starting the process of exchanging the energy with the ball. Let u'_j be the last time the j -th particle collided with the ball before time t_1 . Then for some small neighborhood $U_j \subset U_{j-1}$ of $\Gamma(t_1)$, some $c_j > 0$ and $\mathbf{z}_{n+1} = (\mathbf{z}_j, \mathbf{z}_{j+}) \in U_j$,

$$\begin{aligned} \mathbb{P}_n(\mathbf{Z}_j(t_1) \in d\mathbf{z}_j \mid \mathbf{Z}_{n+1}(u_j)) / d\mathbf{z}_j &\geq c_j, \\ \mathbb{P}_n(\mathbf{Z}_{n+1}(t_1) \in d\mathbf{z}_{n+1} \mid \mathbf{Z}_{n+1}(u'_j)) / d\mathbf{z}_{n+1} \\ &\geq c_j \mathbb{P}_n(\mathbf{Z}_{(j+1)+}(t_1) \in d\mathbf{z}_{(j+1)+} \mid \mathbf{Z}_{n+1}(u'_j)) / d\mathbf{z}_{(j+1)+}. \end{aligned}$$

We apply this claim to $j = n-1$ to obtain

$$\mathbb{P}_n(\mathbf{Z}_{n+1}(t_1) \in d\mathbf{z}_{n+1} \mid \mathbf{Z}_{n+1}(u'_{n-1})) / d\mathbf{z}_{n+1} \geq c_1 \mathbb{P}_n(\mathbf{Z}_{n+}(t_1) \in d\mathbf{z}_{n+} \mid \mathbf{Z}_{n+1}(u'_n)) / d\mathbf{z}_{n+}.$$

At this point it remains to analyze the interaction of the n -th particle and the ball. The position of the ball and its energy and velocity direction can be all made to have a joint density by collisions with the n th point particle after time u_n .

Finally, the position and velocity direction of the n -th point particle have a conditional density given everything else, because of its Lambertian reflections from the bottom of the container. The energy of the n -th particle cannot be adjusted but this is fine because the same energy conservation principle applies to systems starting from other initial conditions. $\square$

Acknowledgements. We are grateful to Shuntao Chen, Persi Diaconis, Martin Hairer, Robert Holyst, Werner Krauth, Mathew Penrose and David Ruelle for the most useful advice. We thank the anonymous referees for suggestions for improvement.

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