

ASYMPTOTIC DIRECTIONS OF BROWNIAN MOTION AT DOMAIN EXITS

KRZYSZTOF BURDZY

ABSTRACT. We study the angular cluster set of Brownian motion at its first exit from a domain. In the plane, the circle–semicircle dichotomy holds at exits on non-singleton complementary components, and hence for countably connected domains. It also holds for Denjoy domains and for exits on countable unions of lines and circles. The assertion for arbitrary planar domains reduces to complements of compact totally disconnected nonpolar sets. In dimensions $d \geq 3$, we prove the sphere–hemisphere dichotomy for bounded uniform domains, where the pathwise cluster set agrees with the geometric directional cluster set of the domain. Exterior-cone points satisfy the hemisphere conclusion in arbitrary domains. Without geometric assumptions, a cone estimate bounds the first spherical Dirichlet eigenvalue of every open region containing the cluster set in terms of the Hausdorff dimension of harmonic measure. Spherical symmetrization then gives a positive surface-measure bound; in dimension three, the explicit universal fraction is $(1 - 1/\sqrt{3})/2$. We also give a local-dimension refinement and criteria for directions to be visible to a minimal harmonic transform.

1. HUMAN COMMENT

The proof has been generated by ChatGPT. The human author did not check the correctness of the proof.

2. INTRODUCTION

Let $D \subset \mathbb{R}^d$ be a domain, $d \geq 2$, and let X be standard Brownian motion started at $x_0 \in D$. Write

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\}, \quad Z = X_{\tau_D} \quad \text{on } \{\tau_D < \infty\}.$$

The set of asymptotic directions at exit is

$$(2.1) \quad A_D(X) = \bigcap_{\varepsilon > 0} \overline{\left\{ \frac{X_t - Z}{|X_t - Z|} : 0 \leq t < \tau_D, \tau_D - t < \varepsilon \right\}} \subset \mathbb{S}^{d-1}.$$

Equivalently, $v \in A_D(X)$ if and only if there are times $t_n < \tau_D$, $t_n \uparrow \tau_D$, for which $(X_{t_n} - Z)/|X_{t_n} - Z| \rightarrow v$. These are pathwise sequences; their terms need not be stopping times. Every assertion about exit directions below is understood on the event of finite exit.

We say that D has the *exit-direction property* if, for every starting point, almost surely on finite exit, $A_D(X)$ is either $\mathbb{S}^{d-1}$ or a closed hemisphere

$$H_n = \{v \in \mathbb{S}^{d-1} : v \cdot n \geq 0\}, \quad n \in \mathbb{S}^{d-1}.$$

The orientation may depend on the path. When $d = 2$, we identify $\mathbb{R}^2$ with $\mathbb{C}$ and refer to the circle–semicircle dichotomy.

For $z \in \partial D$, define the geometric directional cluster set by

$$(2.2) \quad K_D(z) = \bigcap_{r > 0} \overline{\left\{ \frac{y - z}{|y - z|} : y \in D \cap B(z, r) \right\}}.$$

Always $A_D(X) \subset K_D(Z)$. For an open angular region $U \subset \mathbb{S}^{d-1}$, write

$$C_U(z) = \{z + rv : r > 0, v \in U\}.$$

We use $\delta_D(y) = \text{dist}(y, \partial D)$. If $q = \mathbb{P}_{x_0}(\tau_D < \infty) > 0$, normalized harmonic measure is

$$\omega(H) = q^{-1} \mathbb{P}_{x_0}(Z \in H, \tau_D < \infty).$$

For bounded domains, $q = 1$. When $q = 0$, assertions about finite exits are vacuous. Green functions are normalized for the Brownian generator $\frac{1}{2}\Delta$.

For simply connected planar domains, the exit-direction property follows from the minimal-fine version of McMillan's theorem [16, 8]. Its higher-dimensional analogue is Problem 1 in [9]. The results below concern both the shape and the size of the angular cluster set.

Section 4 proves the planar dichotomy on non-singleton complementary components and in Denjoy domains, and reduces the general planar assertion to complements of compact totally disconnected sets. Sections 5 and 6 prove angular maximality and the dichotomy for bounded uniform domains in dimensions $d \geq 3$, together with the exterior-cone and localization consequences. Section 7 gives a slit-domain example where angular maximality fails and a necessary condition for a minimal harmonic transform to omit directions.

For arbitrary domains in $\mathbb{R}^d$, $d \geq 3$, let $s = \dim_{\mathbb{H}} \omega$. Section 8 proves that every connected proper open $U \supset A_D(X)$ satisfies, simultaneously almost surely,

$$\lambda_1(U) \leq s(s - d + 2),$$

where λ_1 is the first Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{d-1}}$. In Section 9, spherical symmetrization yields

$$\sigma(A_D(X)) \geq \sigma\{\theta \in \mathbb{S}^{d-1} : \theta_1 > 1/\sqrt{d}\} > 0,$$

where σ is normalized surface measure. If $s \leq d - 1$, the lower bound improves to $1/2$; if $s = d - 2$, the cluster set is the entire sphere. The general area estimates do not determine the shape of the cluster set.

3. LOCALIZATION AND MINIMAL HARMONIC TRANSFORMS

Lemma 3.1. *On every continuous path with finite exit, $A_D(X)$ is nonempty, compact, and connected.*

Proof. The direction process is continuous before exit. The closures of its images on terminal time intervals are nonempty compact connected subsets of $\mathbb{S}^{d-1}$, and these sets are nested as the intervals decrease. $\square$

Lemma 3.2 (Countable localization). *Let $(V_j)_{j \geq 1}$ be domains in $\mathbb{R}^d$ having the exit-direction property. Almost surely, that property holds for every finite exit from D for which there are j and $a < \tau_D$ such that*

$$X_t \in V_j \quad (a < t < \tau_D), \quad Z \in \partial V_j.$$

The same assertion holds with the hemisphere conclusion alone if that conclusion holds for each V_j .

Proof. For each fixed j and nonnegative rational u , the Markov property at u shows that, on $\{X_u \in V_j\}$, the angular cluster set at the first subsequent finite exit from V_j has the asserted form. Intersect these probability-one events over j and u . On a path satisfying the hypothesis, choose rational $u \in (a, \tau_D)$. The first exit from V_j after u is then exactly τ_D . Deleting the initial path segment does not change (2.1). $\square$

Let D now be a Greenian domain and let h be a positive minimal harmonic function in D . Minimality means that every harmonic function v satisfying $0 \leq v \leq h$ is a constant multiple of h . Write $\mathbb{P}_x^h$ for the law of the Doob h -transform and ζ for its lifetime. The Martin representation and conditioning theorem express killed Brownian motion as a mixture of transforms by minimal Martin kernels; see [12, 13].

Lemma 3.3 (Terminal zero-one law). *Under a transform by a minimal positive harmonic function, every event depending only on the path arbitrarily near its lifetime and unchanged by deleting an initial path segment has probability zero or one, independently of the starting point. If the process has finite lifetime and a fixed Euclidean endpoint z , its angular cluster set is a deterministic nonempty compact connected set.*

Proof. For such an event F , put $p(x) = \mathbb{P}_x^h(F)$. The Markov property at exits from relatively compact balls shows that hp is harmonic and $0 \leq hp \leq h$. Minimality gives $p \equiv c$. Let T_m be exit times from an increasing relatively compact exhaustion of D . The same Markov property gives

$$\mathbb{E}_x^h[\mathbf{1}_F \mid \mathcal{F}_{T_m}] = c.$$

Here $T_m \uparrow \zeta$ and $T_m < \zeta$ almost surely, since the transformed process has no killing in the interior. The full path before its lifetime is measurable with respect to $\sigma(\bigcup_m \mathcal{F}_{T_m})$. Martingale convergence gives $\mathbf{1}_F = c$ almost surely.

For the last assertion, apply the zero-one law to visits arbitrarily near the lifetime to the cones $C_U(z)$, where U ranges over a countable basis of spherical caps. The resulting values determine the closed angular cluster set independently of the starting point. Its compactness and connectedness follow from Lemma 3.1. $\square$

4. PLANAR DOMAINS

4.1. The simply connected theorem. The simply connected case follows from the minimal-fine version of McMillan's theorem [16] in Burdzy [8, Theorems 3.1 and 3.4]. Indeed, a Riemann map has, at almost every boundary point of its source, either a finite nonzero minimal-fine derivative or the twist property. In the first case, its first-order expansion along a conditioned Brownian path transfers the semicircle of directions in the source domain. In the second case, the continuous argument oscillates without bound in both directions along every approaching path, and the cluster set is the whole circle. Conformal time changes preserve these cluster sets. We use this simply connected exit-direction property below.

4.2. Non-singleton complementary components. Let $K = \widehat{\mathbb{C}} \setminus D$. For $z \in K$, denote by $C_K(z)$ the connected component of K containing z , and set

$$N_D = \{z \in \partial D : C_K(z) \neq \{z\}\}, \quad S_D = \partial D \setminus N_D.$$

The components are taken in the Riemann sphere, including its point at infinity.

Theorem 4.1. *For every planar domain D and every starting point in D , the circle–semicircle dichotomy holds almost surely on*

$$\{\tau_D < \infty, Z \in N_D\}.$$

We first record the elementary compactness fact used in the proof.

Lemma 4.2. *Let K be a compact subset of a compact metric space, and let L be closed. The union of the components of K that meet L is compact.*

Proof. Suppose $z_n \rightarrow z$, where $z_n \in C_n \subset K$ and each C_n is a component meeting L . Passing to a subsequence in the space of nonempty compact subsets, we may assume that C_n converges in Hausdorff distance to a compact connected set $C \subset K$. Choosing $w_n \in C_n \cap L$ and passing to a further subsequence gives a point $w \in C \cap L$. Since $z \in C$, the component of K containing z meets L . $\square$

Proof of Theorem 4.1. Let $\mathcal{L}$ be the countable family of horizontal and vertical lines with rational coordinate. For $\ell \in \mathcal{L}$, put $L = \ell \cup \{\infty\}$ and define

$$F_\ell = L \cup \bigcup \{C : C \text{ is a component of } K, C \cap L \neq \emptyset\}.$$

By Lemma 4.2, F_ℓ is compact. It is connected, because each of the components added to the connected set L meets L . Also $\infty \in F_\ell$ and

$$(4.1) \quad F_\ell \setminus K \subset \ell.$$

Every component V of $\widehat{\mathbb{C}} \setminus F_\ell$ is a simply connected plane domain. To see this directly, a Jordan curve in V cannot separate two points of the connected set F_ℓ . Since $\infty \in F_\ell$, the bounded interior of the Jordan curve is disjoint from F_ℓ and therefore belongs to V . The Jordan-curve criterion for simple connectivity applies. There are at most countably many such components. Consequently, the components arising from all $\ell \in \mathcal{L}$ form one countable family of simply connected domains.

Consider a path exiting at $z \in N_D$. Choose a rectangle with rational coordinates whose interior contains z and whose closure does not contain all of $C_K(z)$. Connectedness forces $C_K(z)$ to meet a side of the rectangle. The full line ℓ containing this side belongs to $\mathcal{L}$, does not contain z , and meets $C_K(z)$. Thus $z \in F_\ell$.

Since $z \notin \ell$, a terminal segment of the path avoids ℓ . That segment lies in D , so (4.1) shows that it avoids all of F_ℓ . By connectedness it lies in a single component V of $\widehat{\mathbb{C}} \setminus F_\ell$. Its limit z belongs to ∂V . Lemma 3.2 now proves the assertion. $\square$

Corollary 4.3. *Every countably connected planar domain has the exit-direction property. More generally, the conclusion holds whenever the set S_D of singleton complementary components has harmonic measure zero; in particular, it holds if S_D is polar.*

Proof. In a countably connected domain, S_D is countable. A countable planar set is polar and is almost surely not hit by Brownian motion started outside it. Apply Theorem 4.1 to the remaining exits. $\square$

4.3. Denjoy domains. A Denjoy domain is a domain $\Omega = \mathbb{C} \setminus E$, where E is a proper closed subset of $\mathbb{R}$. We assume that E is nonpolar; otherwise there are almost surely no finite exits. Write $H^+ = \{\operatorname{Im} w > 0\}$ and $H^- = \{\operatorname{Im} w < 0\}$.

Theorem 4.4. *Every Denjoy domain has the exit-direction property. The semicircle alternative, when it occurs, is the upper or lower closed semicircle.*

Here are the precise results on Denjoy domains that we use. Fix a finite boundary point $z \in E$. The positive harmonic functions associated with z form a cone $\mathcal{P}_z$: its elements are bounded outside each neighborhood of z and vanish at all regular boundary points other than z . The boundary condition at infinity is imposed only when infinity is regular. This is the convention in Carroll and Gardiner [10, p. 271]. The relevant Martin-boundary facts of Ancona [2] and Benedicks are summarized there and in Bishop [6, pp. 631–632]:

- (1) $\mathcal{P}_z$ has either one minimal generator or two linearly independent minimal generators.
- (2) In the one-generator case, that generator is invariant under reflection in $\mathbb{R}$.
- (3) In the two-generator case, there is $u \in \mathcal{P}_z$ such that

$$(4.2) \quad \liminf_{t \downarrow 0} t u(z + it) > 0.$$

The last two assertions are the finite-pole versions of [5, Theorem 3]. Bishop's approach theorem gives a further almost-everywhere assertion: at harmonic-measure almost every point with one generator, the conditioned process hits both $(z - r, z)$ and $(z, z + r)$ almost surely, for every $r > 0$ [6, Theorem 1 and its accompanying discussion]. Irregular points have harmonic measure zero.

The following consequence of Lemma 3.3 connects a Poisson atom with the side from which the conditioned path approaches its endpoint.

Lemma 4.5. *Suppose that h is minimal in Ω . If its restriction to H^+ has a positive Poisson atom at $z \in E$, then its transform has finite lifetime, converges to z , and eventually stays in H^+ almost surely. The analogous assertion holds for H^- .*

Proof. Write the half-plane Poisson representation as

$$h(w) = b \operatorname{Im} w + \int_{\mathbb{R}} \frac{\operatorname{Im} w}{|w - s|^2} \mu(ds), \quad w \in H^+,$$

absorbing the normalizing constant into μ . If $a = \mu(\{z\}) > 0$, the half-plane conditioning formula gives

$$\mathbb{P}_w^h \{\zeta < \infty, X_{\zeta-} = z, X_t \in H^+ \text{ for all } 0 \leq t < \zeta\} \geq \frac{a \operatorname{Im} w}{|w - z|^2 h(w)} > 0, \quad w \in H^+.$$

Indeed the Poisson atom represents paths that exit the half-plane at z , before any exit through another point of its boundary. Such paths also remain in Ω up to their terminal time. The event of having finite lifetime and endpoint z , and eventually remaining in H^+ , is a terminal event. Lemma 3.3 raises its positive probability to one. $\square$

Proof of Theorem 4.4. Disintegrate killed Brownian motion over the minimal Martin boundary [13]. For almost every kernel in this disintegration, the transformed process has finite lifetime and a finite Euclidean endpoint. The exceptional Euclidean points in Bishop's theorem have zero mass in this disintegration.

First suppose that the cone at the endpoint z has one generator h . By Lemma 3.3, the angular cluster set is a deterministic compact connected set A_h , independent of the starting point. Since Ω and h are invariant under reflection in $\mathbb{R}$, so is A_h . Bishop's theorem implies $1, -1 \in A_h$: visits to the two intervals at arbitrarily small radii must occur arbitrarily near the lifetime. A proper compact connected subset of the circle is a closed arc. No such arc containing both 1 and -1 is invariant under complex conjugation. Hence $A_h = \mathbb{S}^1$.

Now suppose that the cone has two minimal generators h_1, h_2 . For any positive harmonic function v on H^+ , its Poisson atom at z is

$$a_+(v) = \lim_{t \downarrow 0} t v(z + it).$$

This follows directly from the Poisson representation; the contribution from its nonatomic part tends to zero. In particular, the atom is linear in v . Decomposing the function u in (4.2) into h_1, h_2 shows that at least one generator has a positive upper atom. Applying the same argument to $u(\bar{w})$ gives a generator with a positive lower atom.

A positive atom implies almost sure eventual residence in the corresponding half-plane, by Lemma 4.5. The same minimal generator cannot have positive atoms on both sides, since the two eventual-residence events are disjoint. Thus the two generators are exhausted by the upper and lower alternatives.

Returning to the original Brownian motion, almost every path in the two-generator case eventually stays in H^+ or H^- . On this event, its terminal time is the first exit from that half-plane after some rational time. Countable localization to the two half-planes shows that its cluster set is the corresponding closed semicircle. $\square$

Corollary 4.6. *Let D be any planar domain and let Γ be a countable union of lines and circles. The exit-direction dichotomy holds almost surely on $\{\tau_D < \infty, Z \in \Gamma\}$.*

Proof. For a fixed line ℓ , put $E = (\mathbb{C} \setminus D) \cap \ell$. If E is polar there is nothing to prove. If $E = \ell$, the domain D lies in one of its half-planes and localization applies. Otherwise $D \subset \mathbb{C} \setminus E$, a rotated Denjoy domain. On the event $Z \in E$, the first exit times from these two domains agree, so their cluster sets agree. Apply Theorem 4.4.

For a circle contained in D , the assertion is vacuous. Otherwise use a Möbius transformation with pole at a fixed point of that circle outside D . The pole is almost surely not the exit point. On each remaining finite-exit path, the conformal time change is finite and the derivative at the exit

point is finite and nonzero. The transformation therefore rotates the cluster set and transfers the line case. Finally take the countable union. $\square$

4.4. Reduction to totally disconnected compact sets.

Theorem 4.7. *The following assertions are equivalent:*

- (1) *Every planar domain has the exit-direction property.*
- (2) *For every compact totally disconnected nonpolar set $F \subset \mathbb{C}$, the complement $\mathbb{C} \setminus F$ has the exit-direction property.*

Proof. Only the second implication needs proof. Suppose that D has a positive-probability set of finite exits violating the dichotomy. By Theorem 4.1, these exits lie in S_D almost surely.

The set S_D is Borel. Indeed, with respect to any compatible metric on $\widehat{\mathbb{C}}$, the sets

$$\{z \in K : \text{diam } C_K(z) \geq 1/n\}$$

are closed: a Hausdorff limit of connected compact subsets of K is connected, and diameter passes to the limit. Thus the singleton component locus is a G_δ subset of K .

Define the finite Borel measure

$$\nu(B) = \mathbb{P}_{x_0}\{\tau_D < \infty, Z \in B, A_D(X) \text{ is neither a circle nor a semicircle}\}.$$

The angular cluster set is a measurable compact-set-valued function of the stopped path, so this definition is legitimate. By inner regularity there is a compact $F \subset S_D$ with $\nu(F) > 0$. Every connected subset of F lies in a component of K , and hence is a singleton. Thus F is totally disconnected. It is nonpolar because Brownian motion hits it with positive probability.

We have $D \subset \mathbb{C} \setminus F$. On $\{Z \in F\}$, the first hitting time of F is exactly τ_D . The same paths therefore violate the exit-direction property for $\mathbb{C} \setminus F$ with positive probability, contrary to (2). The complement of a compact totally disconnected planar set is connected; equivalently, such a set cannot separate the plane. One could also apply (2) componentwise without using this elementary topological fact. $\square$

Remark 4.8. Theorem 4.7 isolates an uncountable singleton-component case. It cannot be discarded on grounds of polarity: a compact Cantor set on a line may be nonpolar. Theorem 4.4 handles such sets when they lie on a line, but the reduction imposes no such geometric restriction on F .

5. UNIFORM DOMAINS

A domain is *uniform* if there is $C \geq 1$ such that any $x, y \in D$ can be joined by a rectifiable curve $\gamma \subset D$ satisfying

$$\ell(\gamma) \leq C|x - y|, \quad \text{dist}(w, \partial D) \geq C^{-1} \min\{\ell(\gamma[x, w]), \ell(\gamma[w, y])\}, \quad w \in \gamma.$$

Theorem 5.1. *Let $d \geq 3$ and let $D \subset \mathbb{R}^d$ be a bounded uniform domain. Then, for Brownian motion started at any $x_0 \in D$,*

$$(5.1) \quad A_D(X) = K_D(Z) \quad \text{almost surely.}$$

Moreover, almost surely,

$$A_D(X) = \mathbb{S}^{d-1} \quad \text{or} \quad A_D(X) = \{v \in \mathbb{S}^{d-1} : v \cdot n \geq 0\}$$

for some unit vector n .

O'Neill [17, p. 23] announced the sphere-hemisphere dichotomy for three-dimensional nontangentially accessible (NTA) domains. His paper [18] gives a geometric and stochastic proof of the planar twist-point theorem. Theorem 5.1 requires only interior uniformity; exterior corkscrews and local finiteness of surface measure are not assumed.

5.1. Conditioning and interior-ball estimates. In this subsection, let $D \subset \mathbb{R}^d$, $d \geq 3$, be bounded; uniformity is not required. Let G_D be its Brownian Green function, and let h be a positive minimal harmonic function normalized by $h(x_0) = 1$. We use the transform notation of Section 3.

By the Martin representation and conditioning theorem, the law of killed Brownian motion is the mixture of the transforms by minimal Martin kernels, with mixing measure the representing measure of the constant function 1. For almost every kernel in this mixture, the lifetime is finite and there is a Euclidean boundary limit. These last assertions follow by disintegration from the corresponding properties of Brownian motion in a bounded domain. We refer to [12, 13] for the conditioning theory.

Lemma 5.2 (Hitting an interior ball). *Let $\overline{B}(y, 4\rho) \subset D$ and $x_0 \notin \overline{B}(y, \rho)$. If T is the first hitting time of $\overline{B}(y, \rho)$, then*

$$(5.2) \quad \mathbb{P}_{x_0}^h(T < \zeta) \geq a_d h(y) G_D(x_0, y) \rho^{d-2},$$

where $a_d > 0$ depends only on d .

Proof. The strong Markov property for the Green function gives

$$G_D(x_0, y) = \mathbb{E}_{x_0}[G_D(X_T, y); T < \tau_D].$$

This identity follows by applying the strong Markov property to occupation times in $B(y, \rho)$. The free-space Green bound on $\partial B(y, \rho)$ implies

$$G_D(x_0, y) \leq C_d \rho^{2-d} \mathbb{P}_{x_0}(T < \tau_D).$$

The transform identity and the interior Harnack inequality now give

$$\mathbb{P}_{x_0}^h(T < \zeta) = \mathbb{E}_{x_0}[h(X_T); T < \tau_D] \geq a'_d h(y) \mathbb{P}_{x_0}(T < \tau_D).$$

Combining the inequalities proves the claim. $\square$

Proposition 5.3 (Visibility criterion). *Suppose that under $\mathbb{P}_{x_0}^h$ the process has finite lifetime and converges to a fixed $z \in \partial D$. Let $U \subset \mathbb{S}^{d-1}$ be an open cap. Let $y_j \rightarrow z$ and $\overline{B}(y_j, 4\rho_j) \subset D$, and assume*

$$\overline{B}(y_j, \rho_j) \subset C_U(z), \quad \limsup_{j \rightarrow \infty} h(y_j) G_D(x_0, y_j) \rho_j^{d-2} > 0.$$

Then the process visits $C_U(z)$ at times arbitrarily close to its lifetime, almost surely.

Proof. Pass to a subsequence along which the products are bounded below by a positive constant. Lemma 5.2 gives a common positive lower bound for the probabilities of hitting these balls. Therefore the probability of hitting infinitely many of them is positive:

$$\mathbb{P}_{x_0}^h(\limsup_j \{T_j < \zeta\}) \geq \limsup_j \mathbb{P}_{x_0}^h(T_j < \zeta) > 0.$$

Since $z \in \partial D$, $4\rho_j < |y_j - z|$ and the balls converge to z . A compact initial segment of a path in D can meet only finitely many of them. Thus the event of hitting infinitely many is unchanged by deleting an initial path segment. Lemma 3.3 makes its probability one, and the hitting times must approach ζ . $\square$

5.2. Boundary Harnack estimates and angular maximality. We now assume that $D \subset \mathbb{R}^d$ is bounded and uniform, with $d \geq 3$. We write $f \asymp g$ when f/g is bounded above and below by positive constants independent of the points and scales under consideration. We use two results of Aikawa [1, Theorems 1 and 3, Corollary 3]: the scale-invariant boundary Harnack principle, valid for locally bounded positive harmonic functions vanishing quasi-everywhere on the relevant boundary portion, and the identification of the minimal Martin boundary with ∂D . Thus, with $h_z(x_0) = 1$,

$$(5.3) \quad h_z(y) = \lim_{w \rightarrow z, w \in D} \frac{G_D(y, w)}{G_D(x_0, w)}.$$

The law conditioned on $Z = z$ is the h_z -transform for harmonic measure almost every z .

Uniformity also supplies interior corkscrew balls and Harnack chains. In particular, for all sufficiently small r and $z \in \partial D$ there is $a \in D \cap B(z, r)$ with $\delta_D(a) \geq cr$. Points at distance $O(r)$ from each other and at distance at least $a_0 r$ from the boundary can be joined by a bounded number of overlapping interior balls of radii comparable to r . The constants may depend on a_0 and the uniformity constant. These properties follow directly from the defining curves: the length condition localizes the curve, and the boundary-distance condition, together with the endpoint balls, gives its thickness.

Lemma 5.4 (Green–Martin product). *For each $a_0 > 0$ there are positive constants c, C, r_0 such that*

$$(5.4) \quad cr^{2-d} \leq G_D(x_0, y)h_z(y) \leq Cr^{2-d}$$

whenever $z \in \partial D$, $r = |y - z| < r_0$, and $\delta_D(y) \geq a_0 r$.

Proof. Let A_0 be the enlargement constant in the boundary Harnack principle. Fix $\eta > 0$ so small that $A_0\eta < 1/4$, and choose an interior corkscrew point a for $B(z, \eta r)$, so

$$|a - z| < \eta r, \quad \delta_D(a) \geq c_0 \eta r.$$

For sufficiently small r , both poles y, x_0 are outside $B(z, A_0 \eta r)$. Boundary Harnack applied to $G_D(y, \cdot)$ and $G_D(x_0, \cdot)$ in this ball, followed by (5.3), gives

$$(5.5) \quad h_z(y) \asymp \frac{G_D(y, a)}{G_D(x_0, a)}.$$

Both functions have the required boundedness and quasi-everywhere zero boundary values, since their poles lie outside the enlarged ball.

The Harnack chain between a and y , localized in $B(z, C_1 r)$, avoids x_0 if r_0 is small enough. Consequently

$$(5.6) \quad G_D(x_0, a) \asymp G_D(x_0, y).$$

Moreover,

$$(5.7) \quad G_D(y, a) \asymp r^{2-d}.$$

The upper estimate follows from the free-space bound and $|y - a| \geq (1 - \eta)r$. For the lower estimate, the Green function of $B(y, \delta_D(y))$ gives a lower bound of order r^{2-d} on $\partial B(y, \delta_D(y)/4)$. Starting at a , follow a uniform curve toward y only until its first entry into this smaller ball. The part followed has length $O(r)$, stays a distance at least $c_1 r$ from ∂D , and stays outside $B(y, \delta_D(y)/4)$. A bounded Harnack chain for $G_D(y, \cdot)$ therefore transfers the lower bound to a . Combining (5.5)–(5.7) proves the lemma. $\square$

Lemma 5.5 (Balls in a prescribed angular neighborhood). *Let $z \in \partial D$, let $v \in K_D(z)$, and let U be an open neighborhood of v in $\mathbb{S}^{d-1}$. Then there exist $y_j \rightarrow z$, $r_j = |y_j - z|$, and constants $a, b > 0$ such that*

$$\delta_D(y_j) \geq ar_j, \quad \overline{B}(y_j, 4br_j) \subset D, \quad \overline{B}(y_j, br_j) \subset C_U(z).$$

Proof. Choose $w_j \in D$, $w_j \rightarrow z$, whose directions converge to v , and write $s_j = |w_j - z|$. Fix a small $\varepsilon > 0$, depending on the distance of v from the complement of U . If $\delta_D(w_j) \geq \varepsilon s_j$, use $y_j = w_j$. Otherwise choose a nearest boundary point q_j , so $|q_j - w_j| < \varepsilon s_j$, and an interior corkscrew point $y_j \in B(q_j, \varepsilon s_j)$ with $\delta_D(y_j) \geq c\varepsilon s_j$. In both cases $|y_j - w_j| \leq 2\varepsilon s_j$ and $|y_j - z| \asymp s_j$. For sufficiently small ε , all these directions lie in a smaller neighborhood compactly contained in U , after discarding finitely many indices. A sufficiently small fixed $b > 0$ then gives all three assertions. $\square$

Proposition 5.6 (Angular maximality). *For a bounded uniform domain $D \subset \mathbb{R}^d$, $d \geq 3$,*

$$A_D(X) = K_D(Z) \quad \mathbb{P}_{x_0}\text{-almost surely.}$$

Proof. Fix a boundary point z for which conditioning on $Z = z$ gives the minimal transform h_z with finite lifetime and endpoint z ; this holds for harmonic measure almost every z . If a cap U meets $K_D(z)$, Lemma 5.5 supplies balls in $C_U(z)$. Lemma 5.4 gives

$$h_z(y_j)G_D(x_0, y_j)(br_j)^{d-2} \geq cb^{d-2} > 0.$$

Proposition 5.3 shows that $C_U(z)$ is visited arbitrarily near the lifetime, almost surely under this conditional law. Apply this statement simultaneously to a countable base of spherical caps. Every neighborhood of every $v \in K_D(z)$ is then visited arbitrarily late, so $K_D(z) \subset A_D(X)$. The reverse inclusion is immediate from $X_t \in D$ for $t < \tau_D$. Finally integrate the conditional assertion against harmonic measure. Measurability follows from the same countable base of caps. $\square$

6. EXTERIOR CONES AND LOCALIZATION

Lemma 6.1 (Lipschitz domains). *For a bounded Lipschitz domain $V \subset \mathbb{R}^d$, $d \geq 3$, the directional cluster set of Brownian motion at its first exit is the closed inward hemisphere, almost surely.*

Proof. A bounded connected Lipschitz domain is uniform. Dahlberg's theorem [11] gives absolute continuity of harmonic measure with respect to $\mathcal{H}^{d-1}$ on its boundary. Rademacher's theorem, applied in the boundary charts, therefore gives a differentiable boundary graph at almost every exit point. At such a point z , the geometric cluster set $K_V(z)$ is exactly the closed inward hemisphere: directions strictly to the inward side belong to the domain for all sufficiently short displacements, and strictly outward directions do not. Proposition 5.6 finishes the proof. $\square$

Proposition 6.2. *Let $D \subset \mathbb{R}^d$ be any open set, $d \geq 3$, and start Brownian motion in D . On the event that $\tau_D < \infty$ and Z admits an open truncated cone contained in D^c , the set $A_D(X)$ is a closed hemisphere, almost surely.*

Proof. We first construct a countable family of local Lipschitz overdomains. It suffices to consider cones from a countable family of directions, narrower apertures, and smaller radii. Fix one such choice and rotate coordinates so that its direction is e_d . Write the cone as

$$C = \{(u, t) : t > L|u|, 0 < |(u, t)| < r\}, \quad L > 0,$$

and let $E \subset \partial D$ be the set of vertices z for which $z + C \subset D^c$.

Cover space by countably many open balls $B(c, s)$ with $s < r/8$, and fix one nonempty set $E_0 = E \cap B(c, s)$. If $z, w \in E_0$, then

$$(6.1) \quad |z_d - w_d| \leq L|z' - w'|.$$

Indeed, otherwise one of the two boundary points would lie in the open exterior cone based at the other, at distance less than r , and hence in the interior of D^c , a contradiction.

Define the finite L -Lipschitz function

$$f(u) = \inf_{w \in E_0} \{w_d + L|u - w'|\}, \quad u \in \mathbb{R}^{d-1}.$$

Equation (6.1) gives $f(z') = z_d$ for $z \in E_0$. We also have

$$(6.2) \quad D \cap B(c, 2s) \subset \{(u, t) : t < f(u)\}.$$

For if $x \in D \cap B(c, 2s)$ and $x_d > f(x')$, some $w \in E_0$ satisfies $x_d > w_d + L|x' - w'|$. Since $|x - w| < 3s < r$, this puts x in $w + C$, a contradiction. Equality is excluded by openness of $D \cap B(c, 2s)$.

Choose a bounded Lipschitz domain V agreeing with $\{t < f(u)\}$ on $B(c, 2s)$. One explicit choice is a region $\{u \in Q : -M < t < f(u)\}$, where Q is a sufficiently large cube and $-M$ lies strictly below both the graph on $\overline{Q}$ and the ball in question. This is a bounded Lipschitz domain; the graph, side faces, and bottom meet with a common interior transverse direction in each corner chart. Thus

$$D \cap B(c, 2s) \subset V, \quad E_0 \subset \partial V.$$

Running through the countable cone choices and balls produces countably many such deterministic domains V .

For each of these domains and each nonnegative rational time q , the Markov property and Lemma 6.1 show that Brownian motion started at $X_q \in V$ has hemispherical directions at its first subsequent exit from V , outside a null event. Take the union of these countably many null events. If the original process exits D at $Z \in E_0$, continuity gives a rational $q < \tau_D$ such that $X_t \in B(c, 2s)$ for $q \leq t \leq \tau_D$. By (6.2), the first exit from V after q is precisely τ_D . The desired hemispherical assertion follows. $\square$

Proof of Theorem 5.1. By Proposition 5.6, $A_D(X) = K_D(Z)$ almost surely. If $K_D(Z) = \mathbb{S}^{d-1}$, this equality gives $A_D(X) = \mathbb{S}^{d-1}$. Otherwise the compact set $K_D(Z)$ misses a neighborhood of some direction. Definition (2.2) then supplies an open truncated cone based at Z which does not meet D . Apply Proposition 6.2. $\square$

Corollary 6.3. *Let $D \subset \mathbb{R}^d$ be open, $d \geq 3$, and let (V_j, B_j, E_j) be a countable family, where each V_j is a bounded uniform domain, B_j is open,*

$$D \cap B_j \subset V_j, \quad E_j \subset \partial D \cap \partial V_j \cap B_j.$$

On $\{\tau_D < \infty, Z \in \bigcup_j E_j\}$ the sphere-hemisphere conclusion holds almost surely.

Proof. Use Theorem 5.1 in V_j and the rational-time localization from the proof of Proposition 6.2. $\square$

7. VISIBILITY IN GENERAL DOMAINS

Throughout this section, $d \geq 3$. The equality $A_D(X) = K_D(Z)$ can fail when distinct Martin boundary points correspond to the same Euclidean boundary point.

Example 7.1. Let

$$D = B(0, 3) \setminus \{(u, 0) \in \mathbb{R}^{d-1} \times \mathbb{R} : |u| \leq 1\}, \quad x_0 = (0, \dots, 0, 1).$$

The domain D is connected, and Brownian motion started at x_0 has positive probability of exiting through the relative interior of the removed disk. At every such exit point z ,

$$K_D(z) = \mathbb{S}^{d-1}.$$

A path exiting at z is eventually confined to one side of the disk, since a change of side sufficiently close to z would entail an earlier hit of the disk. Localization to a countable family of half-balls, followed by Lemma 6.1, shows that $A_D(X)$ is almost surely the closed hemisphere corresponding to that side. Thus $A_D(X) \neq K_D(Z)$ on an event of positive probability.

For a minimal harmonic transform with a fixed Euclidean endpoint, the hitting estimate yields a necessary condition for a direction to be absent from the cluster set.

Proposition 7.2. *Let $D \subset \mathbb{R}^d$ be a bounded domain, $d \geq 3$, and let h be a positive minimal harmonic function with $h(x_0) = 1$. Suppose that under $\mathbb{P}_{x_0}^h$ the process has finite lifetime ζ and converges to a fixed $z \in \partial D$. Define $A_D(X)$ as above, with ζ in place of τ_D . There is a deterministic nonempty compact connected set $K_h \subset \mathbb{S}^{d-1}$ such that $A_D(X) = K_h$ almost surely. If U is an open spherical cap and $\bar{U} \cap K_h = \emptyset$, then*

$$(7.1) \quad \lim_{\substack{y \rightarrow z, y \in D \\ (y-z)/|y-z| \in U}} G_D(x_0, y) h(y) \delta_D(y)^{d-2} = 0.$$

Proof. Lemma 3.3 gives the deterministic compact connected set K_h .

Choose a slightly larger spherical cap W such that $\bar{U} \subset W$ and $\bar{W} \cap K_h = \emptyset$. Since $\delta_D(y) \leq |y - z|$, there is a fixed $b \in (0, 1/4)$ such that

$$\bar{B}(y, 4b\delta_D(y)) \subset D, \quad \bar{B}(y, b\delta_D(y)) \subset C_W(z)$$

whenever $(y - z)/|y - z| \in U$. If (7.1) failed, there would be a sequence $y_j \rightarrow z$ in these directions for which

$$\limsup_{j \rightarrow \infty} G_D(x_0, y_j) h(y_j) (b\delta_D(y_j))^{d-2} > 0.$$

Proposition 5.3 would then imply visits to $C_W(z)$ arbitrarily near ζ . Compactness would give a cluster point in $\bar{W}$, contrary to $\bar{W} \cap K_h = \emptyset$. $\square$

8. SPECTRAL RESTRICTIONS IN ARBITRARY DOMAINS

Throughout this section, $D \subset \mathbb{R}^d$ is arbitrary, $d \geq 3$, and $\mathbb{P}_{x_0}(\tau_D < \infty) > 0$. Let ω be the normalized harmonic measure defined in the introduction, and put

$$s = \dim_H \omega := \inf\{\dim_H F : F \subset \partial D \text{ is Borel, } \omega(F) = 1\}.$$

We have $d - 2 \leq s \leq d$. For the lower bound, put $\delta = \text{dist}(x_0, \partial D) > 0$. The usual ball-hitting formula gives, for $z \in \partial D$ and $r < \delta/2$,

$$\omega(B(z, r)) \leq q^{-1} \left(\frac{r}{|x_0 - z|} \right)^{d-2} \leq q^{-1} \delta^{2-d} r^{d-2}.$$

The mass distribution principle now gives $s \geq d - 2$.

Let σ denote normalized surface measure on $\mathbb{S}^{d-1}$.

For an open connected set $U \subset \mathbb{S}^{d-1}$ define

$$\lambda_1(U) = \inf_{0 \neq f \in H_0^1(U)} \frac{\int_U |\nabla_S f|^2 d\sigma}{\int_U |f|^2 d\sigma}, \quad \kappa(U) = \frac{d-2}{2} + \sqrt{\frac{(d-2)^2}{4} + \lambda_1(U)}.$$

The eigenvalue is for $-\Delta_{\mathbb{S}^{d-1}}$, without the Brownian factor $1/2$. In particular,

$$(8.1) \quad \kappa(U)(\kappa(U) - d + 2) = \lambda_1(U).$$

The exponent $\kappa(U)$ governs approach to the vertex of the cone generated by U . This is the same cone characteristic that appears in Le Gall's higher-dimensional cone-point results [14]; see also Bañuelos–Smits [4] for the spectral representation of Brownian motion in cones.

Theorem 8.1. *Almost surely, simultaneously for every proper connected open set $U \subset \mathbb{S}^{d-1}$ containing $A_D(X)$,*

$$(8.2) \quad \kappa(U) \leq s, \quad \text{equivalently} \quad \lambda_1(U) \leq s(s - d + 2).$$

8.1. A cone estimate. Let $U \subset \mathbb{S}^{d-1}$ be proper, connected and open, and let φ be a positive first Dirichlet eigenfunction on U . Its existence follows from the compact embedding of $H_0^1(U)$ into $L^2(U)$; interior elliptic regularity and the strong maximum principle give smoothness and positivity in U . No regularity of ∂U is needed here. For $r > 0$ and $\theta \in U$, define

$$h(r\theta) = r^{-\kappa(U)} \varphi(\theta).$$

The polar-coordinate formula for the Laplacian and (8.1) show that h is positive and harmonic in the cone $C_U = C_U(0)$,

$$C_U = \{r\theta : r > 0, \theta \in U\}.$$

Lemma 8.2. *Let $K \subset U$ be nonempty and compact. There is a finite constant $C = C(U, K)$ such that, for $R > 0$, $w/|w| \in K$, and $|w| > R$,*

$$(8.3) \quad \mathbb{P}_w \left(T_R < \tau_{C_U}, \frac{X_{T_R}}{R} \in K \right) \leq C \left(\frac{R}{|w|} \right)^{\kappa(U)},$$

where $T_R = \inf\{t \geq 0 : |X_t| \leq R\}$.

Proof. Write $\kappa = \kappa(U)$ and let $a = \min_K \varphi > 0$, $b = \max_K \varphi < \infty$. On the event in (8.3), $h(X_{T_R}) \geq aR^{-\kappa}$. Optional stopping for the positive harmonic function h , first on compact subdomains of the cone and with a bounded time horizon, gives

$$aR^{-\kappa} \mathbb{P}_w(T_R < \tau_{C_U}, X_{T_R}/R \in K) \leq h(w) \leq b|w|^{-\kappa}.$$

Fatou's lemma removes the localization. Only interior values of h on the target part of the sphere enter the lower bound, so boundary continuity of the eigenfunction is not required. $\square$

Lemma 8.3. *Fix a deterministic time $u \geq 0$, a number $\rho > 0$, a connected proper open set $U \subset \mathbb{S}^{d-1}$, and a compact set $K_0 \subset U$. Let*

$$E = \left\{ u < \tau_D < \infty, |X_u - Z| \geq \rho, \frac{X_t - Z}{|X_t - Z|} \in K_0 \text{ for every } u \leq t < \tau_D \right\},$$

and define $\nu(H) = \mathbb{P}_{x_0}(E, Z \in H)$. There is a finite constant C , depending on U, K_0, ρ , such that

$$(8.4) \quad \nu(B(z, r)) \leq Cr^{\kappa(U)} \quad (z \in \mathbb{R}^d, r > 0).$$

Proof. Choose a compact neighborhood $K \subset U$ of K_0 . Choose $M > 2$ large enough that whenever $|v - w| \leq 1$ and $|v| \geq M$, the direction $v/|v|$ is in K if $w/|w| \in K_0$. This follows from

$$\left| \frac{v}{|v|} - \frac{w}{|w|} \right| \leq \frac{2|v - w|}{|v|}.$$

Suppose E occurs and $Z \in B(z, r)$, with $r < \rho/(4M)$. Until its first entrance after u into $\overline{B}(z, Mr)$, the path relative to z has direction in K . At time u its distance from z is at least $\rho - r \geq \rho/2 > Mr$. It must enter that ball before τ_D , since $|Z - z| < r$. At this entrance its direction relative to z is still in K .

Thus the event defining $\nu(B(z, r))$ is contained in the event that, after time u , Brownian motion hits the radius- Mr sphere in a direction in K , before leaving the translated cone $z + C_U$. The Markov property at u and Lemma 8.2 give

$$\nu(B(z, r)) \leq C(2Mr/\rho)^{\kappa(U)}.$$

The future endpoint is not conditioned on in this use of the Markov property; only an event inclusion is used. For larger r , enlarge the constant and use $\nu(\mathbb{R}^d) \leq 1$. $\square$

8.2. Proof of the spectral restriction.

Proof of Theorem 8.1. For a fixed U , suppose that $\kappa(U) > s$. There is a deterministic Borel set $F \subset \partial D$ such that $\omega(F) = 1$ and $\dim_H F < \kappa(U)$. In particular, $\mathcal{H}^{\kappa(U)}(F) = 0$. Every measure ν in Lemma 8.3 is carried by F , whereas (8.4) implies that it vanishes on F . Hence $\mathbb{P}_{x_0}(E) = 0$.

Exhaust U by the compact sets

$$K_n = \{\theta \in U : \text{dist}(\theta, \mathbb{S}^{d-1} \setminus U) \geq 1/n\},$$

and take rational u and $\rho = 1/m$, where $m \geq 1$ is an integer. It follows that, almost surely, $A_D(X)$ cannot be contained in this U when $\kappa(U) > s$. Indeed, compactness of $A_D(X)$ would imply that the terminal directions eventually belong to some K_n . A rational u can then be chosen in that terminal interval, and $|X_u - Z| > 0$ permits a choice of m .

We must make this assertion simultaneous in U . Choose a countable basis of geodesic balls on the sphere and let $\mathcal{U}$ be the countable family of proper connected finite unions of these balls. The preceding argument gives a single event of probability one on which no $V \in \mathcal{U}$ with $\kappa(V) > s$ contains $A_D(X)$.

By Lemma 3.1, the set $A_D(X)$ is compact and connected. If $A_D(X) \subset U$ for an arbitrary connected proper open U , cover $A_D(X)$ by finitely many basis balls whose closures lie in U ,

each meeting $A_D(X)$. Their union V is connected, contains $A_D(X)$, and lies in U . Variational monotonicity gives

$$\lambda_1(V) \geq \lambda_1(U), \quad \kappa(V) \geq \kappa(U).$$

Thus $\kappa(U) > s$ would contradict the already simultaneous assertion for $V \in \mathcal{U}$. This proves (8.2) for every U . The equivalence in (8.2) follows from (8.1) and $s \geq d - 2$. $\square$

Remark 8.4 (A local version). The spectral conclusion can be sharpened by replacing s with the lower local dimension of harmonic measure at the exit point,

$$\underline{d}_\omega(Z) = \liminf_{r \downarrow 0} \frac{\log \omega(B(Z, r))}{\log r}.$$

To see this, let ν be any of the restricted measures used in the proof. Since $\nu \ll \omega$, differentiation of measures gives $\nu(B(z, r))/\omega(B(z, r)) \rightarrow d\nu/d\omega(z) > 0$ for ν -almost every z . Estimate (8.4) therefore implies $\underline{d}_\omega(z) \geq \kappa(U)$ for ν -almost every z . The same countable localization and domain monotonicity give, simultaneously for all U containing $A_D(X)$, $\kappa(U) \leq \underline{d}_\omega(Z)$ almost surely.

9. SURFACE MEASURE AND HAUSDORFF DIMENSION

Retain the assumptions and notation of Section 8. For $0 < m < 1$ let B_m be a geodesic cap of normalized area m , and put $\Lambda_d(m) = \lambda_1(B_m)$. The function Λ_d is continuous and strictly decreasing, with limits ∞ and 0 at 0 and 1 ; we extend it by $\Lambda_d(1) = 0$. Here the limit at 1 uses $d - 1 \geq 2$: a point on the sphere has zero H^1 capacity. If $s > d - 2$, define $b_d(s) \in (0, 1)$ by

$$(9.1) \quad \Lambda_d(b_d(s)) = s(s - d + 2),$$

and set $b_d(d - 2) = 1$.

Corollary 9.1. *Almost surely,*

$$(9.2) \quad \sigma(A_D(X)) \geq b_d(s).$$

If $s = d - 2$, then $A_D(X) = \mathbb{S}^{d-1}$ almost surely. If $s \leq d - 1$, then $\sigma(A_D(X)) \geq 1/2$ almost surely.

Corollary 9.2. *For every domain $D \subset \mathbb{R}^d$, almost surely on finite exit,*

$$(9.3) \quad \sigma(A_D(X)) \geq c_d, \quad c_d := \sigma\{\theta \in \mathbb{S}^{d-1} : \theta_1 > 1/\sqrt{d}\} > 0.$$

In particular, $\dim_H A_D(X) = d - 1$. In dimension three,

$$c_3 = \frac{1 - 1/\sqrt{3}}{2} = 0.2113248654 \dots$$

There is also a constant $\eta_d > 0$, independent of D , such that $\sigma(A_D(X)) \geq c_d + \eta_d$ almost surely on finite exit.

The last assertion uses Bourgain's theorem. The explicit bound (9.3) uses only the elementary bound $s \leq d$. The conclusions concern area; they do not assert that $A_D(X)$ contains a spherical cap or a hemisphere.

The spherical Faber–Krahn inequality states that

$$(9.4) \quad \lambda_1(U) \geq \Lambda_d(\sigma(U)).$$

It holds for arbitrary proper open sets. It follows by spherical decreasing rearrangement, which preserves the L^2 norm and decreases Dirichlet energy; see Sperner [19] and the statement in Ashbaugh–Levine [3, §2.C, (2.14)]. Its variational form does not require a smooth boundary.

Proof of Corollary 9.1. The assertion is immediate when $A_D(X) = \mathbb{S}^{d-1}$. Otherwise consider the spherical neighborhoods

$$U_\varepsilon = \{\theta : \text{dist}(\theta, A_D(X)) < \varepsilon\}.$$

For sufficiently small $\varepsilon > 0$ they are connected proper open sets, and their surface measures decrease to $\sigma(A_D(X))$. The simultaneous statement of Theorem 8.1 applies to these random neighborhoods and, together with (9.4), gives

$$(9.5) \quad \Lambda_d(\sigma(U_\varepsilon)) \leq \lambda_1(U_\varepsilon) \leq s(s-d+2).$$

Since $\Lambda_d(m) \rightarrow \infty$ as $m \downarrow 0$, this first shows $\sigma(A_D(X)) > 0$. Letting $\varepsilon \downarrow 0$ gives (9.2) by continuity and strict monotonicity of Λ_d .

If $s = d - 2$ and $A_D(X)$ were proper, a sufficiently small neighborhood U_ε would omit an open cap. Its first Dirichlet eigenvalue would be positive, contradicting (9.5). Finally, a hemisphere has first eigenvalue $d - 1$, with positive eigenfunction $\theta \mapsto \theta_1$. Thus $\Lambda_d(1/2) = d - 1$. When $s \leq d - 1$, we have $s(s-d+2) \leq d - 1$, which gives $\sigma(A_D(X)) \geq 1/2$. $\square$

Proof of Corollary 9.2. The elementary bound $s \leq d$ gives $s(s-d+2) \leq 2d$. The cap

$$Q_d = \{\theta \in \mathbb{S}^{d-1} : \theta_1 > 1/\sqrt{d}\}$$

has first eigenvalue $2d$. Indeed,

$$\varphi(\theta) = \theta_1^2 - 1/d$$

is positive in Q_d , vanishes on its boundary, and satisfies $-\Delta_{\mathbb{S}^{d-1}}\varphi = 2d\varphi$. Positivity identifies its eigenvalue as the first one. Hence $b_d(d) = \sigma(Q_d) = c_d$, proving (9.3). Positive surface measure gives $\dim_{\text{H}} A_D(X) = d - 1$.

For reference, the cap fraction can be written

$$c_d = \frac{\int_{1/\sqrt{d}}^1 (1-t^2)^{(d-3)/2} dt}{\int_{-1}^1 (1-t^2)^{(d-3)/2} dt}.$$

For $d = 3$ this is $(1 - 1/\sqrt{3})/2$.

Bourgain [7] gives a number $\varepsilon_d > 0$, depending only on d , such that $s \leq d - \varepsilon_d$ for every domain. We may decrease it so that $0 < \varepsilon_d < 1$. The function b_d is strictly decreasing, so

$$\eta_d := b_d(d - \varepsilon_d) - b_d(d) > 0, \quad \sigma(A_D(X)) \geq b_d(d - \varepsilon_d) = c_d + \eta_d.$$

$\square$

Remark 9.3 (Comparison with the planar bound). The cone estimate and the countable localization used in Theorem 8.1 also apply when $d = 2$. A proper connected open angular region of length α has first eigenvalue π^2/α^2 and characteristic $\kappa = \pi/\alpha$. If $A_D(X)$ is a proper closed arc of length L , applying the estimate to containing open arcs and letting $\alpha \downarrow L$ gives $sL \geq \pi$, where $s = \dim_{\text{H}} \omega$. This excludes $L = 0$ and gives $L \geq \pi/s$ when $s > 0$; if $s = 0$, the cluster set must be the whole circle. In higher dimensions the spectral restriction controls area rather than determining the shape of the cluster set. In particular, the area estimates do not imply the sphere-hemisphere dichotomy. The half-area conclusion follows from $\dim_{\text{H}} \omega \leq d - 1$, a hypothesis that is unavailable for all domains because of the Wolff snowflake examples; see [15].

REFERENCES

- [1] H. Aikawa, Boundary Harnack principle and Martin boundary for a uniform domain, *J. Math. Soc. Japan* **53** (2001), 119–145. doi:10.2969/jmsj/05310119.
- [2] A. Ancona, Une propriété de la compactification de Martin d'un domaine euclidien, *Ann. Inst. Fourier (Grenoble)* **29** (1979), no. 4, 71–90.

- [3] M. S. Ashbaugh and H. A. Levine, Inequalities for Dirichlet and Neumann eigenvalues of the Laplacian for domains on spheres, *Journées Équations aux dérivées partielles* (1997), Exp. No. I, 15 pp.
- [4] R. Bañuelos and R. G. Smits, Brownian motion in cones, *Probab. Theory Related Fields* **108** (1997), 299–319.
- [5] M. Benedicks, Positive harmonic functions vanishing on the boundary of certain domains in $\mathbb{R}^n$, *Ark. Mat.* **18** (1980), 53–72.
- [6] C. J. Bishop, Brownian motion in Denjoy domains, *Ann. Probab.* **20** (1992), no. 2, 631–651.
- [7] J. Bourgain, On the Hausdorff dimension of harmonic measure in higher dimension, *Invent. Math.* **87** (1987), 477–483.
- [8] K. Burdzy, Minimal fine derivatives and Brownian excursions, *Nagoya Math. J.* **119** (1990), 115–132. doi:10.1017/S0027763000003159.
- [9] K. Burdzy, *My favorite open problems*, Problem 1. https://sites.math.washington.edu/~burdzy/open_mathjax.php. Accessed October 25, 2026.
- [10] T. Carroll and S. J. Gardiner, Lipschitz continuity of the Green function in Denjoy domains, *Ark. Mat.* **46** (2008), 271–283.
- [11] B. E. J. Dahlberg, Estimates of harmonic measure, *Arch. Rational Mech. Anal.* **65** (1977), 275–288. doi:10.1007/BF00280445.
- [12] J. L. Doob, Conditional Brownian motion and the boundary limits of harmonic functions, *Bull. Soc. Math. France* **85** (1957), 431–458. doi:10.24033/bsmf.1494.
- [13] J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart*, Grundlehren der mathematischen Wissenschaften, vol. 262, Springer-Verlag, New York, 1984.
- [14] J.-F. Le Gall, Mouvement brownien, cônes et processus stables, *Probab. Theory Related Fields* **76** (1987), 587–627.
- [15] J. L. Lewis, G. C. Verchota and A. L. Vogel, Wolff snowflakes, *Pacific J. Math.* **218** (2005), no. 1, 139–166. doi:10.2140/pjm.2005.218.139.
- [16] J. E. McMillan, Boundary behavior of a conformal mapping, *Acta Math.* **123** (1969), 43–67.
- [17] M. O’Neill, *Boundary behavior of harmonic functions and Brownian motion*, lecture slides, November 8, 2009. <https://www1.cmc.edu/pages/faculty/MONeill/miscellaneous/main.pdf>.
- [18] M. D. O’Neill, A geometric and stochastic proof of the twist point theorem, *Publ. Mat.* **56** (2012), 41–63. doi:10.5565/PUBLMAT_56112_02.
- [19] E. Sperner, Jr., Zur Symmetrisierung von Funktionen auf Sphären, *Math. Z.* **134** (1973), 317–327.