A BOUNDED BROWNIAN CONCATENATION WITHOUT A BROWNIAN ORIGIN

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1. Human comment

The proof has been generated by ChatGPT. It has been reviewed for correctness and modified by Claude. The human author did not check the correctness of the proof.

2. Introduction and statement

Let $(B^k)_{k\in\mathbb{Z}}$ be independent standard Brownian motions, and let T_k be an almost surely finite stopping time for the natural filtration of B^k. The stopped pieces $(B^k_t)_{0\le t\le T_k}$, $k\in\mathbb{Z}$, are then independent. Suppose that the lengths have divergent sums in both directions, $\sum_{k\ge 0} T_k = \sum_{k<0} T_k = \infty$ almost surely. Set $S_0 = 0$ and $S_{k+1} - S_k = T_k$, and define the continuous concatenation $X = (X_t)_{t\in\mathbb{R}}$ by

$$X_0 = 0, \qquad X_{S_k+t} - X_{S_k} = B^k_t, \quad 0 \le t \le T_k.$$

Write

$$(\theta_s w)(t) = w(s+t) - w(s), \qquad s,t\in\mathbb{R},$$

for a continuous path w, so that $\theta_s\theta_{s'} = \theta_{s+s'}$, and let ν denote the law of a standard two-sided Brownian motion rooted at zero, a probability measure on $C(\mathbb{R})$ with its Borel σ-field. We say that X has a *Brownian origin* if there is an almost surely finite random variable S, possibly defined on an extension of the underlying probability space, such that $\theta_S X$ has law ν.

Burdzy's Problem 8 asks whether a deterministic uniform bound on the lengths T_k guarantees a Brownian origin [1]. When the stopped pieces are identically distributed and have finite mean length, the answer is positive by [2, Theorem 5.1]. This note gives a negative answer without the identical-distribution assumption. Only two stopping rules are used, in a deterministic nonperiodic order, and every length belongs to $[1,2]$; in particular the divergence of the length sums is automatic.

For a continuous path w, define, whenever the indicated window lies in its domain,

$$\Phi_s(w) = \sqrt{30}\int_0^1 (2u-1)w(s+u)\,du. \tag{1}$$

Since the weight integrates to zero, this functional is unchanged when a constant is added to w; it depends only on the increments of w in $[s, s+1]$. Consequently

$$\Phi_s(\theta_c w) = \Phi_{c+s}(w) \qquad \text{for all } c,s\in\mathbb{R}. \tag{2}$$

For a one-sided Brownian motion B, put

$$T(B) = \inf\{t\in[1,2] : \Phi_{t-1}(B) = 0\}\wedge 2, \tag{3}$$

with $\inf\varnothing = \infty$. The quantity $\Phi_{t-1}(B)$ depends only on B restricted to $[t-1,t]$, and $s\mapsto\Phi_s(w)$ is continuous (indeed continuously differentiable, see the proof of Lemma 2). Hence for $t\in[1,2]$ the event $\{T\le t\} = \{\exists\, s\in[1,t] : \Phi_{s-1}(B) = 0\}$ is determined by $(B_u)_{u\le t}$, so T is a stopping time for the natural filtration of B, and $1\le T\le 2$.

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Theorem 1. *Set*

$$b_n = 5n(n+1), \qquad a_n = -(b_n + n), \qquad n \geq 1.$$

For independent Brownian motions $(B^k)_{k\in\mathbb{Z}}$*, choose*

$$T_k = \begin{cases} T(B^k), & k = a_n \text{ for some } n \geq 1, \\ 1, & \text{otherwise}, \end{cases}$$

where T *is given by* (3)*. Then each* T_k *is a stopping time for the natural filtration of* B^k*,* $1 \leq T_k \leq 2$ *for every* k*, and the concatenation* X *has no Brownian origin.*

We call the pieces with index a_n *active*. With $a_0 := 0$ and $b_0 := 0$,

$$a_{n-1} - a_n - 1 = b_n - b_{n-1} = 10n, \qquad n \geq 1, \tag{4}$$

so the indices a_n place $10n$ unit-length pieces between the nth active piece and the next active piece toward the origin; for $n = 1$, the nearer endpoint is the origin itself. These increasing numbers of unit pieces separate the Brownian windows relevant to different active pieces. The stopping rule (3) produces monotone maps from possible starting times to stopping endpoints. The proof first excludes infinite backward orbits for independent copies of these maps (Proposition 4), and then transfers this exclusion to the concatenation (Section 5).

3. Zeros and monotone stopping maps

Under ν, let

$$Y_t = \Phi_{t-1}(W), \qquad \mathcal{Z} = \{t \in \mathbb{R} : Y_t = 0\}.$$

Lemma 2. (i) *Y is a stationary, continuously differentiable, centered Gaussian process with $Y_t \sim N(0,1)$ for every t. In particular $\mathbb{P}(t \in \mathcal{Z}) = 0$ for every fixed $t \in \mathbb{R}$.*

(ii) *Almost surely, all zeros of Y are simple. Hence $\mathcal{Z}$ is a stationary locally finite point process, and its intensity ρ satisfies $\rho \leq \sqrt{10}/\pi$.*

(iii) $0 < q := \mathbb{P}\{\mathcal{Z} \cap (0,1) = \varnothing\} < 1$.

(iv) *For every $a \in \mathbb{R}$, the deterministic path $w_a(t) = (a + \frac{1}{2})t - \frac{1}{2}t^2$ satisfies*

$$\Phi_s(w_a) = \frac{\sqrt{30}}{6}(a - s) \qquad \text{for all } s \in \mathbb{R}. \tag{5}$$

Moreover, for every compact interval J and every $\epsilon > 0$,

$$\mathbb{P}\Big\{ \sup_{s\in J} |\Phi_s(W) - \Phi_s(w_a)| < \epsilon \Big\} > 0. \tag{6}$$

Proof. (i) Let $G(u) = u^2 - u$, so that $G'(u) = 2u - 1$ and $G(0) = G(1) = 0$. Integration by parts gives

$$\Phi_s(W) = -\sqrt{30} \int_0^1 G(u)\, dW_{s+u} = \int_0^1 h(u)\, dW_{s+u}, \qquad h(u) = \sqrt{30}\, u(1-u).$$

Thus Y is a centered Gaussian process, stationary because the increments of W are stationary, with $\operatorname{Var} Y_t = \int_0^1 h(u)^2\, du = 30 \int_0^1 u^2(1-u)^2\, du = 1$. For every continuous w, differentiating (1) under the integral sign after the substitution $t = s + u$ gives

$$\frac{d}{ds}\Phi_s(w) = \sqrt{30}\left(w(s+1) + w(s) - 2\int_s^{s+1} w(t)\, dt \right),$$

which is continuous in s; so Y is continuously differentiable. Integrating by parts once more, $\frac{d}{ds}\Phi_s(W) = -\int_0^1 h'(u)\, dW_{s+u}$, so that $\operatorname{Var} Y_t' = \int_0^1 h'(u)^2\, du = 30\int_0^1 (1-2u)^2\, du = 10$ and $\operatorname{Cov}(Y_t, Y_t') = -\int_0^1 h(u)h'(u)\, du = -\frac{1}{2}\big[h(u)^2\big]_0^1 = 0$. Being jointly Gaussian, Y_t and Y_t' are independent.

(ii) We recall the short argument excluding multiple zeros. Fix a compact interval I and $\delta, \eta > 0$, and let $\omega_\delta = \sup\{|Y'_s - Y'_t| : s, t \in I,\ |s-t| \le \delta\}$. Cover I by at most $|I|/\delta + 2$ mesh points $t_i \in I$ with spacing δ. If $Y_t = Y'_t = 0$ for some $t \in I$ and $\omega_\delta \le \eta$, then the mesh point t_i nearest to t satisfies $|Y_{t_i}| = |\int_t^{t_i} Y'_s\, ds| \le \eta\delta$. Since each Y_{t_i} is standard normal, with density bounded by $(2\pi)^{-1/2}$,

$$\mathbb{P}\{\exists\, t \in I : Y_t = Y'_t = 0\} \le \mathbb{P}\{\omega_\delta > \eta\} + C\eta(|I| + 2\delta).$$

Since Y' is uniformly continuous on I, letting $\delta \downarrow 0$ and then $\eta \downarrow 0$ excludes a simultaneous zero of Y and Y' on I, almost surely. Hence all zeros are simple; being isolated, there are finitely many on each compact interval. Thus $\mathcal{Z}$ is almost surely a locally finite set; it is a random closed set (for instance, $\{\mathcal{Z} \cap I \ne \varnothing\} = \{\inf_{t \in I \cap \mathbb{Q}} |Y_t| = 0\}$ for every compact interval I with rational endpoints, by continuity of Y), and its law is translation invariant by stationarity of Y.

For the intensity, fix a compact interval I. Almost surely, Y is continuously differentiable on I, has finitely many zeros in I, all simple, and $Y \ne 0$ at the two endpoints of I (by (i)). For such a path and all sufficiently small $\epsilon > 0$, the set $\{t \in I : |Y_t| < \epsilon\}$ is a finite union of disjoint intervals, one around each zero, on each of which Y is a diffeomorphism onto $(-\epsilon, \epsilon)$; the change of variables $y = Y_t$ therefore gives

$$\#(\mathcal{Z} \cap I) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_I \mathbf{1}\{|Y_t| < \epsilon\} |Y'_t|\, dt \qquad \text{almost surely.}$$

Fatou's lemma, Fubini's theorem, stationarity, and the independence of Y_0 and Y'_0 yield

$$\begin{aligned}\mathbb{E}\,\#(\mathcal{Z} \cap I) &\le \liminf_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_I \mathbb{E}\big[\mathbf{1}\{|Y_t| < \epsilon\} |Y'_t|\big]\, dt = |I| \lim_{\epsilon \downarrow 0} \frac{\mathbb{P}(|Y_0| < \epsilon)}{2\epsilon}\, \mathbb{E}|Y'_0| \\ &= |I| \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{10}\sqrt{\frac{2}{\pi}} = \frac{\sqrt{10}}{\pi}|I|.\end{aligned}$$

Thus the stationary point process $\mathcal{Z}$ has a finite intensity $\rho \le \sqrt{10}/\pi$. (Rice's formula gives equality, which is not needed.)

(iv) For a continuously differentiable path w, the integration by parts in (i) reads $\Phi_s(w) = \int_0^1 h(u) w'(s+u)\, du$. With $w'_a(t) = a + \frac{1}{2} - t$,

$$\Phi_s(w_a) = \sqrt{30} \int_0^1 u(1-u)\Big(a + \tfrac{1}{2} - s - u\Big)\, du = \sqrt{30}\Big(\frac{a + \frac{1}{2} - s}{6} - \frac{1}{12}\Big) = \frac{\sqrt{30}}{6}(a - s).$$

For continuous paths $w, \tilde{w}$ and every s, (1) gives $|\Phi_s(w) - \Phi_s(\tilde{w})| \le \sqrt{30} \sup_{t \in [s, s+1]} |w(t) - \tilde{w}(t)|$, and Φ_s is unchanged by adding a constant to its argument. Let $J = [c, d]$ and $\epsilon > 0$. The process $(W_t - W_c)_{t \in [c, d+1]}$ has the law of a Brownian motion started at zero, which charges every uniform neighborhood of every continuous path started at zero (the support theorem for Wiener measure). Hence the event

$$\sup_{t \in [c, d+1]} \big|(W_t - W_c) - (w_a(t) - w_a(c))\big| < \epsilon/\sqrt{30}$$

has positive probability, and on this event $\sup_{s \in J} |\Phi_s(W) - \Phi_s(w_a)| < \epsilon$. This proves (6).

(iii) Apply (6) with $J = [-1, 0]$. For $a = 1$, (5) gives $\Phi_s(w_1) \ge \sqrt{30}/6$ on J, so with $\epsilon = \sqrt{30}/6$ we obtain an event of positive probability on which $Y_t = \Phi_{t-1}(W) > 0$ for all $t \in [0, 1]$; hence $q > 0$. For $a = -\frac{1}{2}$, (5) gives $\Phi_{-1}(w_a) = \sqrt{30}/12$ and $\Phi_0(w_a) = -\sqrt{30}/12$, so with $\epsilon = \sqrt{30}/12$ we obtain an event of positive probability on which $Y_0 > 0 > Y_1$; by continuity Y then has a zero in $(0, 1)$, and $q < 1$. □

For a locally finite set $\mathcal{Z} \subseteq \mathbb{R}$ (possibly empty), define

$$f(x) = \min\{\inf(\mathcal{Z} \cap [x+1, \infty)),\ x + 2\}, \qquad x \in \mathbb{R}, \tag{7}$$

with $\inf \varnothing = \infty$. Both functions in the minimum are nondecreasing in x. Thus f is nondecreasing and

$$x + 1 \le f(x) \le x + 2. \tag{8}$$

Since $\mathcal{Z}$ is locally finite, for every x there is $\delta > 0$ with $\mathcal{Z} \cap [x + 1 - \delta, x + 1) = \varnothing$, so $x' \mapsto \inf(\mathcal{Z} \cap [x' + 1, \infty))$ is constant on $(x - \delta, x]$; hence f is left-continuous. Its right limits are

$$f(x+) = \min\{\inf(\mathcal{Z} \cap (x + 1, \infty)),\, x + 2\},$$

so f is continuous at x unless $x+1 \in \mathcal{Z}$: the discontinuities of f lie in $\mathcal{Z}-1$. When $\mathcal{Z}$ is the random set of Lemma 2, the map f built from $\mathcal{Z} - c$ is $x \mapsto f(x + c) - c$, and $\mathcal{Z} - c$ has the law of $\mathcal{Z}$; hence the *displacement field* $x \mapsto f(x) - x$ is a stationary process. We write $\mathbb{P}\{\cdot\}$ and $\mathbb{E}[\cdot]$ for probabilities and expectations under this law.

For the backward-orbit argument we need closed images. Define the correspondence

$$F(x) = \{f(x), f(x+)\}, \qquad F(A) = \bigcup_{x \in A} F(x). \tag{9}$$

At a jump, this includes only the two limiting endpoint values. The graph $\{(x, y) : y \in F(x)\}$ is closed. Indeed, let $(x_i, y_i) \to (x, y)$ with $y_i \in F(x_i)$. If $x_i < x$ for infinitely many i, then along these indices $f(x_i) \le f(x_i+) \le f(x)$ and $f(x_i) \to f(x-) = f(x)$, so $y = f(x)$; if $x_i > x$ for infinitely many i, then along these $f(x+) \le f(x_i) \le f(x_i+)$ and $f(x_i+) \to f(x+)$, so $y = f(x+)$; and if $x_i = x$ for infinitely many i, then $y \in F(x)$ because $F(x)$ is finite. Moreover, $F(A)$ is closed whenever A is closed: if $y_i \in F(x_i)$ with $x_i \in A$ and $y_i \to y$, then $x_i \in [y_i - 2, y_i - 1]$ by (8), so a subsequence of (x_i) converges to some $x \in A$, and the closed graph gives $y \in F(x)$.

Put

$$H = \{y : \mathcal{Z} \cap (y - 1, y) = \varnothing\}. \tag{10}$$

Its complement $H^c = \bigcup_{z \in \mathcal{Z}} (z, z + 1)$ is open, so H is closed; and since $\mathcal{Z}$ is locally finite, every boundary point of this union of open intervals is an endpoint of one of them, so $\partial H \subseteq \mathcal{Z} \cup (\mathcal{Z} + 1)$. Furthermore

$$F(\mathbb{R}) \subseteq \mathcal{Z} \cup H, \qquad y \notin \mathcal{Z},\ y \in F(x) \implies x = y - 2. \tag{11}$$

Indeed, if $\inf(\mathcal{Z} \cap [x + 1, \infty))$ is finite it is attained, because $\mathcal{Z}$ is closed; so $f(x) \notin \mathcal{Z}$ forces $f(x) = x + 2$ and $\mathcal{Z} \cap [x + 1, x + 2] = \varnothing$, whence $f(x) \in H$. Likewise $f(x+) \notin \mathcal{Z}$ forces $f(x+) = x + 2$ and $\mathcal{Z} \cap (x + 1, x + 2] = \varnothing$, whence $f(x+) \in H$. In words, a stopping endpoint outside $\mathcal{Z}$ must come from the cutoff. From (10), stationarity, and $\partial H \subseteq \mathcal{Z} \cup (\mathcal{Z} + 1)$,

$$\mathbb{P}\{y \in H\} = q, \qquad \mathbb{E}\,\#(\partial H \cap I) \le 2\rho|I| \tag{12}$$

for every deterministic y and every bounded interval I.

Lemma 3. *Let $\mathcal{Z}_1, \mathcal{Z}_2, \dots$ be independent copies of the point process $\mathcal{Z}$ of Lemma 2, and let f_n be the map (7) built from $\mathcal{Z}_n$. For every deterministic $x < y$, the two trajectories obtained by successively applying $f_1, f_2, \dots$ to x and to y meet after finitely many steps, almost surely.*

Proof. Let $\mathcal{F}_n = \sigma(\mathcal{Z}_1, \dots, \mathcal{Z}_n)$, $U_0 = x$, $V_0 = y$, $U_{n+1} = f_{n+1}(U_n)$, $V_{n+1} = f_{n+1}(V_n)$, and $D_n = V_n - U_n$. By monotonicity, $D_n \ge 0$, and by (8), $|D_{n+1} - D_n| \le 1$. Since f_{n+1} is independent of $\mathcal{F}_n$ while U_n and V_n are $\mathcal{F}_n$-measurable, the stationarity of the displacement field (applied with the shift U_n) gives, for every bounded measurable $\psi : \mathbb{R}^2 \to \mathbb{R}$,

$$\mathbb{E}\big[\psi\big(f_{n+1}(U_n) - U_n,\ f_{n+1}(V_n) - V_n\big) \,\big|\, \mathcal{F}_n\big] = \Psi(D_n), \qquad \Psi(d) := \mathbb{E}\big[\psi\big(f(0),\, f(d) - d\big)\big]. \tag{13}$$

Taking $\psi(\alpha, \beta) = \beta - \alpha$ shows $\mathbb{E}[D_{n+1} - D_n \mid \mathcal{F}_n] = \mathbb{E}[f(d) - d]\big|_{d = D_n} - \mathbb{E}[f(0)] = 0$, so (D_n) is a nonnegative martingale. Taking $\psi(\alpha, \beta) = (\beta - \alpha)^2$ shows

$$\mathbb{E}\big[(D_{n+1} - D_n)^2 \,\big|\, \mathcal{F}_n\big] = v(D_n), \qquad v(d) := \mathbb{E}\big[(f(d) - f(0) - d)^2\big], \quad d \ge 0.$$

Clearly $v(0) = 0$. The function v is continuous on $[0, \infty)$: by Lemma 2(i), a fixed deterministic point is almost surely outside $\mathcal{Z} - 1$, hence a continuity point of f; so $f(d') - f(0) - d' \to f(d) - f(0) - d$ almost surely as $d' \to d$, and the differences are bounded by 2, so bounded convergence applies.

We also have $v(d) > 0$ for every $d > 0$. Fix $d > 0$, choose $a \in (0, \min(d, \frac{1}{4}))$ and $\delta \in (0, \min(a, d - a))$, and apply (6) with $J = [0, d + 1]$ and $\epsilon = \sqrt{30}\,\delta/6$. By (5), on the resulting event of positive probability, $\Phi_s(W) > 0$ for $s \in [0, a - \delta]$ and $\Phi_s(W) < 0$ for $s \in [a + \delta, d + 1]$; that is, $Y > 0$ on $[1, 1 + a - \delta]$ and $Y < 0$ on $[1 + a + \delta, d + 2]$, and by continuity Y has a zero in $(1 + a - \delta, 1 + a + \delta)$. Hence the first mark in $[1, \infty)$ lies in $(1 + a - \delta, 1 + a + \delta) \subseteq (1, \frac{3}{2})$ (recall $a + \delta < 2a < \frac{1}{2}$), so $f(0) < \frac{3}{2}$, while $[d + 1, d + 2] \subseteq [1 + a + \delta, d + 2]$ contains no mark, so $f(d) = d + 2$. Thus $f(d) - f(0) - d = 2 - f(0) > \frac{1}{2}$ on an event of positive probability, and $v(d) > 0$.

The nonnegative martingale D_n converges almost surely to a finite limit D_∞. For an integer $M > D_0$, let $\tau_M = \inf\{n : D_n \geq M\}$. The stopped martingale $D_{n \wedge \tau_M}$ is bounded by $M + 1$, and by the orthogonality of martingale increments

$$\mathbb{E}\big[D^2_{n\wedge\tau_M}\big] = D_0^2 + \mathbb{E} \sum_{j < n \wedge \tau_M} v(D_j) \leq (M + 1)^2,$$

so $\mathbb{E} \sum_{n < \tau_M} v(D_n) \leq (M + 1)^2$ by monotone convergence. Almost surely the convergent sequence (D_n) is bounded, hence $\tau_M = \infty$ for some integer M, and therefore $\sum_n v(D_n) < \infty$. If $D_\infty > 0$, then D_n eventually lies in the compact interval $[D_\infty/2, 2D_\infty] \subseteq (0, \infty)$, on which the continuous function v is bounded below by a positive constant, contradicting the summability. Hence $D_n \to 0$.

Finally, there is $p > 0$ such that

$$\inf_{0 < d \leq 1/2} \mathbb{P}\{f(0) = f(d)\} \geq p. \tag{14}$$

Choose $a \in (\frac{1}{2}, \frac{3}{4})$, $\delta \in (0, \min(a - \frac{1}{2}, \frac{3}{4} - a))$, and apply (6) with $J = [0, 1]$ and $\epsilon = \sqrt{30}\,\delta/6$. As above, on the resulting event $Y > 0$ on $[1, 1 + a - \delta] \supseteq [1, \frac{3}{2}]$, $Y < 0$ on $[1 + a + \delta, 2]$, and Y has a zero in $(1 + a - \delta, 1 + a + \delta) \subseteq (\frac{3}{2}, \frac{7}{4})$. Let $p > 0$ be the probability of this event. On it, there is no mark in $[1, \frac{3}{2}]$ and the first mark z in $[1, \infty)$ lies in $(\frac{3}{2}, \frac{7}{4})$. For $d \in (0, \frac{1}{2}]$ we have $d + 1 \leq \frac{3}{2}$, so z is also the first mark in $[d + 1, \infty)$, and $z < 2 \leq d + 2$; hence $f(0) = z = f(d)$. This proves (14).

Now fix an integer $N \geq 0$. On the $\mathcal{F}_{N+j}$-measurable event $\{D_{N+j} \in (0, \frac{1}{2}]\}$, (13) and (14) give $\mathbb{P}\{D_{N+j+1} = 0 \mid \mathcal{F}_{N+j}\} \geq p$. Consequently

$$\mathbb{P}\big\{D_{N+j} \in (0, \tfrac{1}{2}] \text{ for } j = 0, 1, \ldots, k\big\} \leq (1 - p)^k \longrightarrow 0 \qquad (k \to \infty).$$

Taking the union over N, almost surely there is no N such that $D_n \in (0, \frac{1}{2}]$ for all $n \geq N$. Since $D_n \to 0$, some N has $D_n \leq \frac{1}{2}$ for all $n \geq N$, and therefore $D_n = 0$ for some n. Once $D_n = 0$ the trajectories coincide forever. □

4. Absence of infinite backward orbits

Proposition 4. *Let $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$ be independent copies of the point process $\mathcal{Z}$ of Lemma 2, and let f_n and F_n be the map* (7) *and the correspondence* (9) *built from $\mathcal{Z}_n$. Almost surely there is no sequence of real numbers $(x_n)_{n \geq 0}$ such that*

$$x_{n-1} \in F_n(x_n) \qquad (n \geq 1). \tag{15}$$

Proof. Step 1: nested closed sets. For $m \geq k \geq 1$, define

$$E_{k,m} = F_k \circ \cdots \circ F_m(\mathbb{R}), \qquad E^{(k)} = \bigcap_{m \geq k} E_{k,m}.$$

Each $E_{k,m}$ is closed, as an iterated image of the closed set $\mathbb{R}$, and $E_{k,m+1} \subseteq E_{k,m}$. For a compact interval I, a point $y \in I$ can only arise as $y \in F_k \circ \cdots \circ F_m(x)$ with $x \in [\inf I - 2(m - k + 1), \sup I - (m - k + 1)]$, by (8), and along such a chain each map is evaluated at points of $[\inf I - 2(m - k + 1), \sup I]$,

where f_j depends only on $\mathcal{Z}_j \cap [\inf I - 2(m-k+1)+1, \sup I + 2]$. Thus $E_{k,m} \cap I$ is determined by the restrictions of $\mathcal{Z}_k, \dots, \mathcal{Z}_m$ to a bounded window; in particular each $E_{k,m}$ is a random closed set, with measurable hit events $\{E_{k,m} \cap I \neq \varnothing\}$. Since the sets $E_{k,m} \cap I$ are compact and decreasing in m, $\{E^{(k)} \cap I \neq \varnothing\} = \bigcap_{m \geq k} \{E_{k,m} \cap I \neq \varnothing\}$, so $E^{(k)}$ is a random closed set as well.

Step 2: the recursion. Compactness of the successive preimage sets gives

$$E^{(k)} = F_k(E^{(k+1)}). \tag{16}$$

The inclusion $\supseteq$ is clear from $E^{(k+1)} \subseteq E_{k+1,m}$ for all m. For $\subseteq$, let $K_m = E_{k+1,m}$, $m \geq k+1$, a decreasing sequence of closed sets, and let $y \in E^{(k)} = \bigcap_{m>k} F_k(K_m)$. Choose $x_m \in K_m$ with $y \in F_k(x_m)$; then $x_m \in [y-2, y-1]$ by (8). A convergent subsequence has a limit x which lies in every K_m (closed and decreasing), that is, $x \in E^{(k+1)}$, and the closed graph of F_k gives $y \in F_k(x)$. This proves (16). Iterating (16) along $k = 1, 2, \dots$ and choosing preimages successively shows that every $x_0 \in E^{(1)}$ is the root of a sequence satisfying (15); conversely, a sequence satisfying (15) has $x_0 \in E_{1,m}$ for every m. Thus $E^{(1)}$ is exactly the set of possible roots x_0 in (15).

Step 3: E is locally finite. Write $E = E^{(1)}$, and let H_k be the set (10) built from $\mathcal{Z}_k$. Consider a sequence (x_n) satisfying (15) with root $y = x_0$. Say that the orbit has a *marked endpoint* in layer j if $x_{j-1} \in \mathcal{Z}_j$. If there is no marked endpoint in layers $1, \dots, k-1$, then (11) gives successively $x_j = x_{j-1} - 2$ and $x_{j-1} \in H_j$ for $j < k$, so $x_j = y - 2j$ for $j \leq k-1$ and $y \in H_j + 2(j-1)$ for $j < k$. Thus either there is a first marked endpoint, or

$$y \in Q_\infty := \bigcap_{k \geq 1} \big(H_k + 2(k-1)\big).$$

We first show that $Q_\infty = \varnothing$ almost surely. For a compact interval $I = [a, b]$ and $m \geq 1$, put $Q_m = \bigcap_{k=1}^m (H_k + 2(k-1))$, a closed set. Suppose $Q_m \cap I \neq \varnothing$ and let $y = \min(Q_m \cap I)$. If $y > a$, then y is the limit of points $y_i \uparrow y$ with $y_i \notin Q_m$; each y_i fails to belong to some $H_{k_i} + 2(k_i - 1)$, some index k occurs infinitely often, and therefore $y \in \partial(H_k + 2(k-1))$ while $y \in H_j + 2(j-1)$ for all $j \neq k$. Hence

$$\mathbb{P}\{Q_m \cap I \neq \varnothing\} \leq \mathbb{P}\{a \in Q_m\} + \sum_{k=1}^{m} \mathbb{E} \sum_{y \in \partial(H_k + 2(k-1)) \cap I} \prod_{j \neq k,\, j \leq m} \mathbf{1}\{y \in H_j + 2(j-1)\}.$$

By independence of the layers and (12), $\mathbb{P}\{a \in Q_m\} = q^m$, and conditioning on $\mathcal{Z}_k$ in the kth term, each point y of $\partial(H_k + 2(k-1)) \cap I$ belongs to the other $m-1$ sets with conditional probability q^{m-1}; so

$$\mathbb{P}\{Q_m \cap I \neq \varnothing\} \leq q^m + 2\rho\, m |I| q^{m-1} \longrightarrow 0 \qquad (m \to \infty), \tag{17}$$

because $q < 1$. Since $Q_\infty \subseteq Q_m$, the set Q_∞ misses I almost surely, and a countable union over intervals with rational endpoints shows $Q_\infty = \varnothing$ almost surely.

Consequently, almost surely every root $y \in E$ has a first marked endpoint, in some layer k, and then $y - 2(k-1) \in \mathcal{Z}_k$ while $y \in H_j + 2(j-1)$ for $j < k$. Hence, almost surely,

$$E \subseteq \mathcal{C} := \bigcup_{k \geq 1} \left[(\mathcal{Z}_k + 2(k-1)) \cap \bigcap_{j<k} (H_j + 2(j-1)) \right].$$

The first-moment (Campbell) formula for the stationary point process $\mathcal{Z}_k + 2(k-1)$, followed by conditioning on $\mathcal{Z}_k$ and independence of the layers, yields for every bounded interval I

$$\mathbb{E}\,\#(\mathcal{C} \cap I) \leq \sum_{k \geq 1} \rho |I|\, q^{k-1} = \frac{\rho |I|}{1-q} < \infty. \tag{18}$$

Hence $\mathcal{C}$, and therefore E, is almost surely locally finite.

Step 4: E is a stationary point process. The joint law of $(\mathcal{Z}_k)_{k\ge1}$ is invariant under a simultaneous translation of all the mark sets, and such a translation by c translates every $E_{k,m}$, hence E, by c. Therefore the random closed set E (Step 1) has a translation-invariant law, and by Step 3 it is almost surely locally finite with $\lambda := \mathbb{E}\,\#(E\cap[0,1]) \le \rho/(1-q) < \infty$. Thus E is a stationary point process of finite intensity λ, and $\mathbb{E}\,\#(E\cap I) = \lambda|I|$ for every bounded interval I.

Step 5: E is the injective image of an independent copy. The tail set $E' = E^{(2)}$ is built from $\mathcal{Z}_2, \mathcal{Z}_3, \ldots$ exactly as E is built from $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$; hence it has the law of E and is independent of $\mathcal{Z}_1$. Conditionally on E', its countably many points are fixed, and by Lemma 2(i) each of them belongs to $\mathcal{Z}_1 - 1$ with probability zero. Thus F_1 is single-valued on E' almost surely, and (16) with $k=1$ reads

$$E = f_1(E') \qquad \text{almost surely.} \tag{19}$$

Now let Ξ have the law of E and be independent of a map f built from a copy of $\mathcal{Z}$. Then (f,Ξ) has the law of (f_1, E'), so $f(\Xi)$ has the law of E by (19). For a bounded interval I, the image counted with multiplicities has expected mass

$$\begin{aligned}\mathbb{E}\sum_{x\in\Xi}\mathbf{1}\{f(x)\in I\} &= \lambda\int_{\mathbb{R}}\mathbb{P}\{f(x)\in I\}\,dx\\ &= \lambda\int_{\mathbb{R}}\mathbb{P}\{x+f(0)\in I\}\,dx = \lambda\,\mathbb{E}\int_{\mathbb{R}}\mathbf{1}\{x+f(0)\in I\}\,dx = \lambda|I|.\end{aligned}$$

Here the first equality is Campbell's formula, valid because f is independent of Ξ, and the second uses stationarity of the displacement field. The image counted without multiplicities, $\#(f(\Xi)\cap I)$, has expected mass $\lambda|I|$ as well, because $f(\Xi)$ has the law of E. Both quantities are finite with the same expectation, and their difference $\sum_{x\in\Xi}\mathbf{1}\{f(x)\in I\} - \#(f(\Xi)\cap I)$ is nonnegative; so it vanishes almost surely. Taking a countable cover of $\mathbb{R}$ by bounded intervals shows that f is injective on Ξ almost surely.

Step 6: conclusion. Start from Ξ as in Step 5 and iterate independent maps $f_1, f_2, \ldots$, also independent of Ξ, setting $\Xi_0 = \Xi$ and $\Xi_n = f_n(\Xi_{n-1})$. By induction using (19), each Ξ_{n-1} has the law of E and is independent of f_n; so by Step 5, f_n is injective on Ξ_{n-1} almost surely, for every $n \ge 1$. On the event that Ξ contains at least two points, choose two of them measurably (for instance, the two points nearest to the origin, the left one first in case of a tie), and call them $x < y$. Conditionally on Ξ, these are deterministic and the maps $f_1, f_2, \ldots$ are independent copies of (7), so Lemma 3 shows that the trajectories of x and y merge at some finite step n, almost surely; at that step f_n is not injective on Ξ_{n-1}. This has probability zero, so $\mathbb{P}\{\#\Xi \ge 2\} = 0$: the stationary point process Ξ contains at most one point on the whole line. Then $\lambda|I| = \mathbb{E}\,\#(\Xi\cap I) \le 1$ for arbitrarily long intervals I, so $\lambda = 0$, and consequently $E = \varnothing$ almost surely. □

5. Proof of Theorem 1

Each T_k in the theorem is a stopping time for the natural filtration of B^k (Section 1), and $1 \le T_k \le 2$. It remains to show that X has no Brownian origin. We first make precise the constraint that a root of the decomposition imposes on a path.

Definition 5. Let $w \in C(\mathbb{R})$. For $A \in \mathbb{R}$ put

$$T_A(w) = T(\theta_A w) = \inf\{t \in [1,2] : \Phi_{A+t-1}(w) = 0\} \wedge 2,$$

where the second equality is (2). A real number r is a *root* of w if there are real numbers $A_1 > A_2 > \cdots$ such that, with $A_0 := r$,

$$A_{n-1} = A_n + T_{A_n}(w) + 10n \qquad (n \ge 1). \tag{20}$$

Let $\mathcal{R} \subseteq C(\mathbb{R})$ denote the set of paths that have a root.

In words, r is a root of w if w admits a decomposition into consecutive pieces with the lengths prescribed by Theorem 1, rooted at r: the nth active piece, counting into the past, occupies $[A_n, A_n + T_{A_n}(w)]$, its length being the stopping rule (3) evaluated on that piece of w, and it is followed by $10n$ unit pieces (recall (4)). Only the active pieces constrain the path; the unit pieces impose nothing, and the pieces with nonnegative index impose nothing either. Note that a root need not determine the times A_n.

To remove the deterministic lengths, write

$$x_0 = r, \qquad x_n = A_n + b_n \quad (n \geq 1), \tag{21}$$

so that (20) and (4) become

$$x_{n-1} = x_n + T_{x_n - b_n}(w) \qquad (n \geq 1), \tag{22}$$

and, since $T_A(w) \in [1, 2]$,

$$x_n = r - \sum_{j=1}^{n} T_{A_j}(w) \in [r - 2n,\, r - n] \qquad (n \geq 1). \tag{23}$$

For $n \geq 1$ and $w \in C(\mathbb{R})$ let

$$\mathcal{Z}_n^w = \{t \in \mathbb{R} : \Phi_{t-b_n-1}(w) = 0\}.$$

By definition of T_A, for every $x \in \mathbb{R}$,

$$x + T_{x-b_n}(w) = \min\big\{ \inf\big(\mathcal{Z}_n^w \cap [x+1, x+2]\big),\, x+2\big\} = \min\big\{ \inf\big(\mathcal{Z}_n^w \cap [x+1, \infty)\big),\, x+2\big\}, \tag{24}$$

because $x - b_n + t - 1 \in (\mathcal{Z}_n^w - b_n - 1)$ if and only if $x + t \in \mathcal{Z}_n^w$. In other words, whenever $\mathcal{Z}_n^w$ is locally finite, the right-hand side of (22) is the map (7) built from $\mathcal{Z}_n^w$, evaluated at x_n.

Lemma 6. *For every $s \in \mathbb{R}$ and every outcome, $-s$ is a root of $\theta_s X$.*

Proof. Put $A_n = S_{a_n} - s$ for $n \geq 1$ and $A_0 = -s = S_0 - s$. By (4), between the active pieces a_n and a_{n-1} there are exactly $10n$ unit pieces, so $A_{n-1} - A_n = S_{a_{n-1}} - S_{a_n} = T_{a_n} + 10n$. It remains to check $T_{a_n} = T_{A_n}(\theta_s X)$. Since $\theta_{A_n}\theta_s X = \theta_{S_{a_n}} X$, and this path coincides with $B := B^{a_n}$ on $[0, T_{a_n}]$ by construction, it suffices to show that $T(u) = T(B)$ for every continuous u that coincides with B on $[0, \tau]$, $\tau := T(B)$. For $t \in [1, \tau]$, the window $[t-1, t]$ lies in $[0, \tau]$, so $\Phi_{t-1}(u) = \Phi_{t-1}(B)$. If $\tau < 2$, then $\Phi_{\tau-1}(B) = 0$ and $\Phi_{t-1}(B) \neq 0$ for $t \in [1, \tau)$; the same holds for u, so $T(u) = \tau$. If $\tau = 2$, then $\Phi_{t-1}(B) \neq 0$ for $t \in [1, 2)$, hence also $\Phi_{t-1}(u) \neq 0$ for $t \in [1, 2)$, so $T(u) = 2$. (This is the statement that T is a stopping time: its value is determined by the path up to time T.) □

Proposition 7. *There is a Borel set $\mathcal{N} \subseteq C(\mathbb{R})$ with $\nu(\mathcal{N}) = 0$ and $\mathcal{R} \subseteq \mathcal{N}$. That is, a path of law ν almost surely has no root.*

Proof. Let W have law ν. Fix an integer $M \geq 1$ and let $\mathcal{R}_M$ be the set of paths with a root in $[-M, M]$. By (23), a root $r \in [-M, M]$ has $x_n \in [-M-2n, M-n]$ for all $n \geq 1$. By (24), to evaluate $x + T_{x-b_n}(W)$ for every $x \in [-M-2n, M-n]$ it suffices to know $\mathcal{Z}_n^W \cap [-M-2n+1, M-n+2]$, that is, the values $\Phi_{t-b_n-1}(W)$ for $t \in [-M-2n+1, M-n+2]$, and these depend only on the increments of W on

$$L_n(M) = [-b_n - 2n - M,\, -b_n - n + M + 2]. \tag{25}$$

The interval $L_{n+1}(M)$ lies to the left of $L_n(M)$, and the gap between its right endpoint and the left endpoint of $L_n(M)$ is

$$(-b_n - 2n - M) - (-b_{n+1} - n - 1 + M + 2) = b_{n+1} - b_n - n - 1 - 2M = 9n + 9 - 2M.$$

Choose $N = N(M)$ such that $9n + 9 - 2M > 0$ for all $n > N$. Then the intervals $L_n(M)$, $n > N$, are pairwise disjoint, so the increment processes of W on these intervals are independent.

Let $(W^{(n)})_{n>N}$ be independent two-sided Brownian motions defined on a probability space $(\Omega', \mathbb{P}')$, and work on the product space $C(\mathbb{R}) \times \Omega'$ with the measure $\nu \otimes \mathbb{P}'$, where W is the first coordinate;

thus W and $(W^{(n)})_{n>N}$ are independent. For $n > N$ define the path $\widehat{W}^{(n)}$ by $\widehat{W}^{(n)}_t = W_{t-b_n} - W_{\ell_n - b_n}$ for $t \in L_n(M) + b_n = [-2n - M,\, -n + M + 2]$, where $\ell_n = \min L_n(M) + b_n$, and by attaching the increments of $W^{(n)}$ continuously outside this interval: $\widehat{W}^{(n)}_t = \widehat{W}^{(n)}_{\ell_n} + W^{(n)}_t - W^{(n)}_{\ell_n}$ for $t < \ell_n$, and similarly to the right of $\max L_n(M) + b_n$. The increments of $\widehat{W}^{(n)}$ on $L_n(M) + b_n$ are those of W on $L_n(M)$, and outside they are those of $W^{(n)}$; hence $(\widehat{W}^{(n)})_{n>N}$ are independent two-sided Brownian motions up to additive constants, which do not affect (1). Let $\widehat{\mathcal{Z}}_n = \{t : \Phi_{t-1}(\widehat{W}^{(n)}) = 0\}$, and let $\widehat{f}_n$ and $\widehat{F}_n$ be the map (7) and the correspondence (9) built from $\widehat{\mathcal{Z}}_n$. By Lemma 2, $(\widehat{\mathcal{Z}}_n)_{n>N}$ are independent copies of $\mathcal{Z}$, so Proposition 4, applied after reindexing, shows that the event

$$\widetilde{G}_M = \Big\{\text{every } \widehat{\mathcal{Z}}_n \ (n > N) \text{ is locally finite, and there is } (x_n)_{n \ge N} \\ \text{with } x_{n-1} \in \widehat{F}_n(x_n) \text{ for all } n > N\Big\}$$

is measurable with $\nu \otimes \mathbb{P}'(\widetilde{G}_M) = 0$. Let Ω_1 be the event that every $\widehat{\mathcal{Z}}_n$, $n > N$, is locally finite; $\nu \otimes \mathbb{P}'(\Omega_1) = 1$ by Lemma 2(ii).

Since Φ is unchanged by additive constants, $\Phi_{t-1}(\widehat{W}^{(n)}) = \Phi_{t-b_n-1}(W)$ whenever $[t-1, t] \subseteq L_n(M) + b_n$, i.e. for $t \in [-M - 2n + 1, M - n + 2]$. Hence $\widehat{\mathcal{Z}}_n \cap [x+1, x+2] = \mathcal{Z}^W_n \cap [x+1, x+2]$ for all $x \in [-M - 2n, M - n]$, and (24) gives, for every outcome,

$$x + T_{x-b_n}(W) = \widehat{f}_n(x) \qquad \text{for all } x \in [-M - 2n, M - n],\ n > N. \tag{26}$$

Now suppose $W \in \mathcal{R}_M$, with root $r \in [-M, M]$ and times A_n as in Definition 5, and let x_n be given by (21). Then $x_n \in [-M - 2n, M - n]$ by (23), and (22) with (26) gives $x_{n-1} = \widehat{f}_n(x_n) \in \widehat{F}_n(x_n)$ for all $n > N$. Thus

$$\{W \in \mathcal{R}_M\} \cap \Omega_1 \subseteq \widetilde{G}_M. \tag{27}$$

By Fubini's theorem, the functions $\phi(w) = \mathbb{P}'\{\omega' : (w, \omega') \in \widetilde{G}_M\}$ and $\phi_1(w) = \mathbb{P}'\{\omega' : (w, \omega') \in \Omega_1\}$ are Borel on $C(\mathbb{R})$ with $\int \phi \, d\nu = 0$ and $\int \phi_1 \, d\nu = 1$. Hence $\mathcal{N}_M := \{\phi > 0\} \cup \{\phi_1 < 1\}$ is a Borel set with $\nu(\mathcal{N}_M) = 0$. If $w \in \mathcal{R}_M$, then (27) gives $\phi(w) \ge \phi_1(w)$, so $w \in \mathcal{N}_M$. Therefore $\mathcal{R}_M \subseteq \mathcal{N}_M$, and $\mathcal{N} := \bigcup_{M \ge 1} \mathcal{N}_M$ is a ν-null Borel set containing $\mathcal{R} = \bigcup_M \mathcal{R}_M$. □

Proof of Theorem 1. Suppose that S is an almost surely finite random variable, possibly defined on an extension of the probability space, such that $\theta_S X$ has law ν. The map $(s, w) \mapsto \theta_s w$ is continuous, so $\theta_S X$ is a random element of $C(\mathbb{R})$. By Lemma 6, $\theta_S X \in \mathcal{R} \subseteq \mathcal{N}$ on the event $\{S \text{ is finite}\}$, with $\mathcal{N}$ the ν-null Borel set of Proposition 7. Hence $\mathbb{P}\{\theta_S X \in \mathcal{N}\} = 1$, whereas $\mathbb{P}\{\theta_S X \in \mathcal{N}\} = \nu(\mathcal{N}) = 0$. This contradiction shows that X has no Brownian origin. □

Remark 8.
(a) The proof shows more than stated. For every almost surely finite random time S, the law of $\theta_S X$ gives full mass to the Borel set $\mathcal{N}$, which is ν-null; so the law of $\theta_S X$ and ν are mutually singular.
(b) The specific sequence $b_n = 5n(n+1)$ enters only through $b_{n+1} - b_n - n \to \infty$, which makes the windows (25) eventually disjoint for every fixed M. Gaps growing at least linearly are needed for this argument, because the set (23) of possible positions of the nth active piece has length n.
(c) Only the inequality $q < 1$ of Lemma 2(iii) is used, in (17) and (18); the inequality $q > 0$ plays no role.

References

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