A Simple Proof of Tychonoff's Theorem
Via Nets

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1. INTRODUCTION. The Tychonoff theorem, a central theorem of point-set topology, states that the product of any family of compact spaces is compact. The current textbook literature contains three standard proofs of this theorem, all of which may be found in the classic text of Kelley [8]: the proof using Alexander's subbase theorem [8, Ch. 5, Th. 6, Th. 13]; the Bourbaki proof using ultrafilters [8, pp. 143–144]; and (at least implicitly) the proof using universal nets [8, p. 81, Ex. J].

Of these, the Bourbaki proof is the most popular; it can be presented very briefly without explicit mention of the theory of filters (cf. [5], [8]). However, it is difficult to motivate without a thorough study of filters. (See Munkres [10, pp. 229–234] for a very thorough elementary motivation of the Bourbaki proof.)

The aim of this note is to present a simple proof of Tychonoff's theorem (new, so far as I know) using only the basic theory of nets together with a straightforward application of Zorn's lemma.

For the convenience of readers who may not be familiar with the net theory of convergence in topological spaces, the next section summarizes the facts we need.

The paper concludes with a few brief comments on the literature.

2. OUTLINE OF THE THEORY OF NETS. The topology of a metric space $M$ is described by the sequences in $M$. In particular, $M$ is compact provided that every sequence of points in $M$ has a subsequence that converges in $M$. But one must generalize the notion of sequence to get a theory of convergence that is adequate for arbitrary topological spaces. The modern theory of generalized sequences, or nets, is due to Kelley [6]. Everything we need is proved in his book [8].

A directed set is a partially ordered set $(A, \preceq)$ such that, given $\alpha$ and $\beta \in A$, there is some $\gamma \in A$ with $\alpha \preceq \gamma$.

Example 1. The positive integers $\mathbb{N}$, directed by the usual order.

Example 2. Let $X$ be a topological space, $p \in X$, and let $\mathcal{N}_p$ be the set of all neighborhoods of the point $p$. For $U, V \in \mathcal{N}_p$ let $U \preceq V$ mean $V \subseteq U$. Then $\mathcal{N}_p$ is directed by reverse inclusion.

A net in a topological space $X$ is a function $x : A \rightarrow X$, where $A$ is any directed set. One says that the net $x$ is based on $A$. Useful notation: write $x(\alpha)$ as $x_\alpha$, and denote the net by $(x_\alpha : \alpha \in A)$. This notation makes nets resemble sequences; of course a sequence is simply a net based on the directed set $\mathbb{N}$.

The net $(x_\alpha : \alpha \in A)$ converges to a point $p \in X$ provided that, given any neighborhood $U$ of $p$, there is some $\alpha \in A$ such that, for all $\beta \geq \alpha$, $x_\beta \in U$. (The limit $p$ is unique if $X$ is a Hausdorff space.) Easy consequence: a subset $S$ of $X$ is closed if and only if the limit of any convergent net of points of $S$ is also in $S$. This shows that nets are indeed adequate to describe the topology of $X$.

A point $q \in X$ is a cluster point of the net $(y_\beta : \beta \in A)$ provided that, given any neighborhood $U$ of $q$ and any $\alpha \in A$, there is some $\beta \geq \alpha$ with $y_\beta \in U$. Example: given a sequence, suppose that $q$ is a limit of some subsequence; then $q$ is a cluster point of the original sequence.

The most subtle concept in the theory is that of a subnet. First, consider two directed sets $A$ and $B$. A map $\phi : B \rightarrow A$ is cofinal provided that, given $\alpha \in A$, there exists $\beta \in B$ so that, for every $\beta' \geq \beta$, we have $\phi(\beta') \preceq \alpha$. Now let $x = (x_\alpha : \alpha \in A)$ be a net in $X$ based on $A$. If $\phi : B \rightarrow A$ is a cofinal map, then the composition $x \circ \phi = (x_{\phi(\alpha)} : \beta \in B)$ is a net based on $B$; we say that $x \circ \phi$ is a subnet of the net $x$.

The following result is important because it relates cluster points to subnets.

Proposition. A point $p$ in $X$ is a cluster point of a net $x$ if and only if there is a subnet of $x$ which converges to $p$.

Finally, we require the characterization of compactness in terms of nets.

Theorem. A topological space $X$ is compact if and only if every net in $X$ has a subnet which converges in $X$. Equivalently, every net in $X$ has a cluster point.

3. PROOF OF TYCHONOFF'S THEOREM. Let $(X_\alpha : \alpha \in A)$ be an indexed family of compact topological spaces. We may assume that these spaces are all non-empty. Recall that the product $\Pi_{\alpha \in A} X_\alpha$ consists of all functions $f$ defined on the index set $I$, such that, for each $i \in I$, $f(i) \in X_i$. A basic neighborhood $N$ of $f$ in the product topology is determined by a finite subset $F \subseteq I$, together with neighborhoods $U_j$ of $f(j)$ in $X_j$ for each $j \in F$; $N$ consists of all $h \in X$ such that, for all $j \in F$, $h(j) \in U_j$. It will be convenient to say that $N$ is supported on $F$, and to write $N = N(U_j : j \in F)$.

By a partially defined member $g$ of the product $X$ we mean a function $g$ with domain $J \subseteq I$, such that, for all $i \in J$, $g(i) \in X_i$. (That is, $g \in \Pi_{i \in J} X_i$.)

Let $(f_\alpha : \alpha \in A)$ be a net in the product space $X$. Suppose that $g$, with domain $J \subseteq I$, is a partially defined member of $X$. Then we say that $g$ is a partial cluster point of the given net provided that, given $\alpha \in A$, for every finite set $F \subseteq J$ and every basic neighborhood $N(U_j : j \in F)$ of $g$ in $\Pi_{i \in J} X_i$, there exists $\beta \in A$, $\beta \geq \alpha$, such that, for all $j \in F$, $f_\beta(j) \in U_j$. (In other words, $g$ is a cluster point in $\Pi_{i \in J} X_i$ of the net $(f_\alpha : i \in J, \alpha \in A)$.) If $g$ has domain $J = I$, then $g$ is a cluster point in $X$ of the net $(f_\alpha : \alpha \in A)$. Our aim is to show the existence of such a $g$, using Zorn's lemma.

To this end, let $\mathcal{P}$ be the set of all partial cluster points of the given net $(f_\alpha : \alpha \in A)$. Note that $\mathcal{P}$ is non-empty because the empty function $\varnothing \in \mathcal{P}$. Partially order $\mathcal{P}$ by inclusion (extension of functions). That is, $g_1 \subseteq g_2$ provided that the domain of $g_1$ is contained in that of $g_2$, and $g_1$ agrees with $g_2$ on their common domain.

Suppose that $\mathcal{L} = \{g_\lambda : \lambda \in \Lambda\}$ is a linearly ordered subset of $\mathcal{P}$. Define $g_\Lambda = \bigcup_{\lambda \in \Lambda} g_\lambda$. Then $g_\Lambda$ is a partially defined member of $X$, because any two members of $\mathcal{L}$ agree on their common domain. Moreover, $g_\lambda \subseteq g_\Lambda$, i.e., $g_\lambda$ is a partial cluster point of the net $(f_\alpha : \alpha \in A)$. This is immediate from the fact that every basic neighborhood of $g_\lambda$ has finite support $F$, and so $F$ is contained in the domain of $g_\lambda$ for some $\lambda \in \Lambda$, and this $g_\lambda$ is a partial cluster point. Accordingly

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\[ g_0 \in \mathcal{P} \text{ and } g_0 \text{ is an upper bound for } \mathcal{L}. \text{ Thus } \mathcal{P} \text{ satisfies the hypothesis of Zorn's lemma.}

Therefore \( \mathcal{P} \) contains a maximal member \( g \). We assert that the domain \( J \) of \( g \) is all of \( I \). If this is not the case choose \( k \in I \setminus J \). Now \( g \) is a cluster point in \( \Pi_{E_J} X_e \) of the net \( \{ f_\alpha \mid J; \alpha \in A \} \) and therefore \( g \) is the limit of some subnet \( \{ f_{\sigma(n)}(k); \beta \in B \} \) has a cluster point \( p \in X_k \). Define a function \( h \) with domain \( J \cup \{ k \} \) by setting \( h - g \) on \( J \) and \( h(k) = p \). Then it is clear that \( h \) is a partial cluster point of the net \( \{ f_\alpha; \alpha \in A \} \), so that \( h \in \mathcal{P} \) and \( h \) is strictly larger than \( g \). This contradicts the maximality of \( g \) in \( \mathcal{P} \). Hence the domain of \( g \) is \( I \), \( g \) is a cluster point of the net \( \{ f_\alpha; \alpha \in A \} \), and the proof that \( X \) is compact is done.

4. COMMENTS ON THE LITERATURE. Tychonoff [12] originally proved that an arbitrary product of compact intervals is compact. The general theorem is due to Čech [4, p. 830]. The “Bourbaki” ultrafilter proof is given by H. Cartan [3]. A form of the “universal net” proof is in Tukey’s thesis [11, p. 36, p. 75]; the modern version is Kelley’s [6].

All proofs of the general Tychonoff theorem involve some form of the axiom of choice: this follows from Kelley’s well-known result [7]. In [9] P. Loeb carefully discusses the role of the axiom of choice, and presents a fairly straightforward proof of Tychonoff’s theorem which avoids the axiom of choice in certain special cases.

REFERENCES


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