Sobolev Stability of Plane Wave Solutions to the NLSE

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Statement of Problem

- Nonlinear Schrödinger Equation

$$i\partial_t u = \Delta u + \lambda |u|^{2p} u \qquad (1)$$

$$x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}$$

- Nonlinear Schrödinger Equation

$$i\partial_t u = \Delta u + \lambda |u|^{2p} u \qquad (1)$$

$$x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}$$

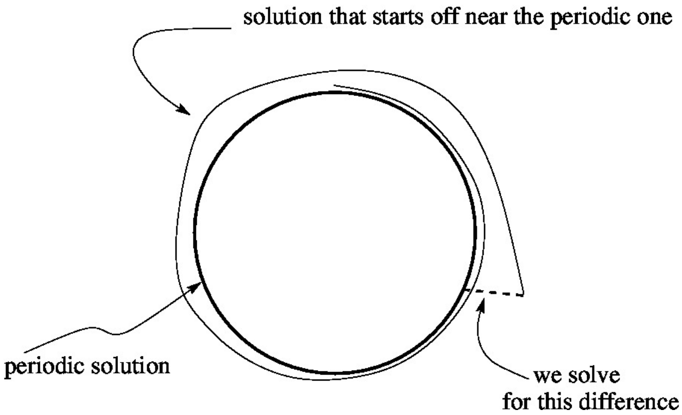
- Consider the plane wave solution to (1):

$$w_m(x,0) := \varrho e^{im\cdot x}$$

$$w_m(x,t) = \varrho e^{im\cdot x} e^{i(|m|^2 - \lambda \varrho^{2p})t}$$

- Assuming $u(x,t)$ satisfies (1) and $\|\varrho - e^{-im\cdot x} u(x,0)\|_{H^s(\mathbb{T}^d)} < \varepsilon$, what type of stability can we expect?

Picture


solution that starts off near the periodic one
periodic solution
we solve
for this difference

Our Goal

Definition (Orbital Stability)

A solution $x(t)$ is said to be orbitally stable if, given $\varepsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any other solution, $y(t)$, satisfying $|x(t_0) - y(t_0)| < \delta$, then $d(y(t), O(x_0, t_0)) < \epsilon$ for $t > t_0$.

- For any $M \in \mathbb{N}$
- There exist s_0 and ε_0 so that for any solution u to (1) with $\|\varrho - e^{-im\cdot x}u(x,0)\|_{H^s(\mathbb{T}^d)} < \varepsilon$, for $\varepsilon < \varepsilon_0$ and $s > s_0$
- $$\inf_{\varphi\in\mathbb{R}} \|e^{-i\varphi}e^{-im\cdot\bullet}w_m(\bullet, t) - e^{-im\cdot\bullet}u(\bullet, t)\|_{H^s(\mathbb{T}^d)} < \varepsilon C(M, s_0, \varepsilon_0)$$
- For $t < \varepsilon^{-M}$.

First Approach

- Assume $m = 0$
- Translation of (1) by w_0:

$$i\partial_t u = (\Delta + (p+1)\lambda\varrho^{2p})u + (p\lambda\varrho^{2(p-1)})w_0^2\bar{u} + \sum_{i=2}^{2p+1} F_i(u, \bar{u}, w_0) \tag{2}$$

$$i\partial_t u_n = (-|n|^2 + (p+1)\lambda\varrho^{2p})u_n + (p\lambda\varrho^{2(p-1)})w_0^2\bar{u}_{-n} + F(u_k, \bar{u}_k, w_0) \tag{3}$$

- The linear part of (3) is a system with periodic coefficients, so we consider Floquet's theorem.

Floquet's Theorem

Theorem (Floquet's Theorem)

Suppose $A(t)$ is periodic. Then the Fundamental matrix of the linear system has the form

$$\Pi(t, t_0) = P(t, t_0) \exp((t - t_0)Q(t_0))$$

where $P(\cdot, t_0)$ has the same period as $A(\cdot)$ and $P(t_0, t_0) = \mathbb{1}$.

The eigenvalues of $M(t_0) := \Pi(t_0 + T, t_0)$, ρ_j, are known as Floquet multipliers and

Corollary

A periodic linear system is stable if all Floquet multipliers satisfy $|\rho_j| \leq 1$.

Constant coefficients and Diagonalization

With $z_n = e^{-i\lambda\varrho^{2p}t}u_n$, the linear part of (3) is

$$i\partial_t \begin{pmatrix} z_n \\ \bar{z}_{-n} \end{pmatrix} = A_n \begin{pmatrix} z_n \\ \bar{z}_{-n} \end{pmatrix}$$

We then diagonalize

$$i\partial_t \begin{pmatrix} x_n \\ \bar{x}_{-n} \end{pmatrix} = \begin{pmatrix} \Omega_n & 0 \\ 0 & \Omega_{-n} \end{pmatrix} \begin{pmatrix} x_n \\ \bar{x}_{-n} \end{pmatrix}$$

where

$$\Omega_n = \sqrt{|n|^2(|n|^2 + 2p\varrho^{2p})}$$

assuming $\lambda = -1$.

Duhamel Iteration Scheme

Duhamel's Formula:

$$x_n(t) = e^{i\Omega_n t} x_n(0) + \int_0^t e^{i\Omega_n (t-s)} F(x(s))_n \, ds$$

Define the iteration scheme:

$$\begin{cases} x_n(t, k+1) = x_n(t, 0) + \int_0^t e^{i\Omega_n(t-s)} F(x_n(s, k)) \, ds \\ x_n(t, 0) := e^{i\Omega_n t} x_n(0, 0) \end{cases}$$

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- This approach is similar to the 19th century approach of expanding the solution in a perturbative series:

  $$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots$$

  u_k being defined recursively.
- This series does not converge, so we should expect a similar phenomenon.

Appearance of small divisors

The first step shows us the issues that this iteration scheme presents us:

Small Model of First Iterate

$$
\begin{aligned}
x_n(t,1) &= x_n(t,0) + \int_0^t e^{i\Omega_n(t-s)} \sum_{n_1,n_2} x_{n_1}(s,0)x_{n_2}(s,0)\,ds \\
&= x_n(t,0) + e^{i\Omega_n t} \sum_{n_1,n_2} x_{n_1}x_{n_2} \int_0^t e^{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)s}\,ds \\
&= x_n(t,0) + \sum_{n_1,n_2} x_{n_1}x_{n_2} \frac{e^{i(\Omega_{n_1}+\Omega_{n_2})t} - e^{i\Omega_n t}}{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)}
\end{aligned}
$$

Recall that

$$\Omega_n = \sqrt{|n|^2(|n|^2 + 2p\varrho^{2p})}$$

and note the pattern

$$\partial_\varrho \Omega_n = \frac{C(n,\varrho)}{\sqrt{|n|^2 + 2p\varrho^{2p}}} = \Omega_n \frac{\tilde{C}(n,\varrho)}{|n|^2 + 2p\varrho^{2p}}$$

$$\partial_\varrho^2 \Omega_n = \frac{-C^2(n,\varrho)}{(|n|^2 + 2p\varrho^{2p})^{3/2}} = \Omega_n \frac{-\tilde{C}^2(n,\varrho)}{(|n|^2 + 2p\varrho^{2p})^2}$$

We can conclude that

$$\Omega_{n_1} + \Omega_{n_2} - \Omega_n = \partial_\varrho(\Omega_{n_1} + \Omega_{n_2} - \Omega_n) = \partial_\varrho^2(\Omega_{n_1} + \Omega_{n_2} - \Omega_n) = 0$$

does not occur on compact set of ϱ.

Appearance of small divisors

Small Model of Second Iterate

$$
\begin{aligned}
&x_n(t,2) \\
&= x_n(t,0) + \int_0^t e^{i\Omega_n(t-s)} \sum_{n_1,n_2} x_{n_1}(s,1) x_{n_2}(s,1)\, ds \\
&= x_n(t,1) \\
&+ e^{i\Omega_n t} \sum_{n_1,k_1,k_2} \frac{x_{n_1} x_{k_1} x_{k_2} \int_0^t e^{i(\Omega_{n_1}+\Omega_{k_1}+\Omega_{k_2}-\Omega_n)s} - e^{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)s}\, ds}{i(\Omega_{k_1}+\Omega_{k_2}-\Omega_{n_2})} \\
&+ e^{i\Omega_n t} \sum_{j_1,j_2,k_1,k_2} \frac{x_{j_1} x_{j_2} x_{k_1} x_{k_2} \int_0^t e^{i(\Omega_{j_1}+\Omega_{j_2}+\Omega_{k_1}+\Omega_{k_2}-\Omega_n)s} - \ldots\, ds}{-(\Omega_{j_1}+\Omega_{j_2}-\Omega_{n_1})(\Omega_{k_1}+\Omega_{k_2}-\Omega_{n_2})} \\
&+ \ldots
\end{aligned}
$$

Issues

- Convergence
- Resonances
- Type of stability
  - Problem at zero mode

A Reduction on the Hamiltonian

$$H := \sum_{k\in\mathbb{Z}^d} |k|^2|u_k|^2 + \frac{1}{p+1} \sum_{\sum_{i=1}^{p+1} k_i = \sum_{i=1}^{p+1} h_i} u_{k_1} \dots u_{k_{p+1}} \bar{u}_{h_1} \dots \bar{u}_{h_{p+1}}. \tag{4}$$

Let $L := \|u(0)\|_{L^2}^2$, define the symplectic reduction of u_0:

$$\{u_k, \bar{u}_k\}_{k\in\mathbb{Z}^d} \to (L, \nu_0, \{v_k, \bar{v}_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}),$$

$$u_0 = e^{i\nu_0}\sqrt{L - \sum_{k\in\mathbb{Z}^d} |v_k|^2}, \quad u_k = v_k e^{i\nu_0}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.$$

A Reduction on the Hamiltonian

$$H = \frac{1}{p+1} L^{p+1} + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (|k|^2 + pL^p)|v_k|^2 + L^p\Big(\frac{p}{2}(v_k v_{-k} + \bar{v}_k \bar{v}_{-k})\Big)$$

$$+ L^{p-\frac{1}{2}} \sum_{\substack{k_1,k_2\in\mathbb{Z}^d\setminus\{0\}\\ k_1+k_2\neq 0}} \Big(\frac{p(p-1)}{6}(v_{k_1} v_{k_2} v_{-k_1-k_2} + c.c) + \frac{(p+1)p}{2}(v_{k_1} v_{k_2} \bar{v}_{k_1+k_2} + c.c.)\Big)$$

$$+ \big(-pL^{p-1} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |v_k|^2\big)\Big(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} (p+1)|v_k|^2 + \frac{p}{2}(v_k v_{-k} + \bar{v}_k \bar{v}_{-k})\Big)$$

$$+ \frac{p}{2} L^{p-1}\big(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |v_k|^2\big)^2$$

$$+ L^{p-1} \sum_{\substack{k_i\in\mathbb{Z}^d\setminus\{0\}\\ k_1+k_2\neq k_3+k_4}} \Big(\frac{p^2(p+1)}{4}(v_{k_1} v_{k_2} \bar{v}_{k_3} \bar{v}_{k_4} + c.c) + \frac{(p+1)p(p-1)}{6}(v_{k_1} v_{k_2} v_{k_3} \bar{v}_{k_4} + c.c.)\Big)$$

$$+ L^{p-1}\Big(\frac{p(p-1)(p-2)}{12} \sum_{\substack{k_i\in\mathbb{Z}^d\setminus\{0\}\\ k_1+k_2\neq k_3+k_4}} (v_{k_1} v_{k_2} v_{k_3} v_{k_4} + c.c)\Big) + h.o.t.$$

We now diagonalize the quadratic part of the Hamiltonian:

$$H_0 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} (k^2 + L^p p)|v_k|^2 + L^p \frac{p}{2}(v_k v_{-k} + \bar{v}_k \bar{v}_{-k}) \qquad (5)$$

which gives

$$H_0 = \sum_{k \in \mathbb{Z}^d} \frac{\Omega_k}{2}(|x_k|^2 + |x_{-k}|^2) \qquad (6)$$

with $\Omega_k = \sqrt{|k|^2(|k|^2 + 2pL^p)}$.

- It is convenient to group together the modes having the same frequency i.e. to denote

$$\omega_q := \sqrt{q^2(q^2 + 2pL^p)}, \qquad q \geq 1. \qquad (7)$$

A fork in the road

- KAM Theory
  - Existence of quasi periodic after a small perturbation that exists for all time
- Birkhoff Normal Forms
  - Orbital ε-stability of the periodic solution up to time ε^{-M}.

Birkhoff Normal Form Theorem in Finite Dimension

Definition (Normal Form)

Let $H = H_0 + P$ where $P \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$, which is at least cubic such that P is a perturbation of H_0. We say that P is in **normal form** with respect to H_0 if it Poisson commutes with H_0:

$$\{P, H_0\} = 0$$

Definition (Nonresonance)

Let $r \in \mathbb{N}$. A frequency vector, $\omega \in \mathbb{R}^n$, is **nonresonant up to order r** if

$$k \cdot \omega := \sum_{j=1}^{n} k_j \omega_j \neq 0 \text{ for all } k \in \mathbb{Z}^n \text{ with } 0 < |k| \leq r$$

Birkhoff Normal Form Theorem in Finite Dimension

Theorem (Moser '68)

Let $H = H_0 + P$ *where*

- $H_0 = \sum_{j=1}^{N} \omega_j \frac{p_j^2 + q_j^2}{2}$
- $P \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ *having a zero of order 3 at the origin*

Fix $M \geq 3$ *an integer. There exists* $\tau : \mathcal{U} \ni (q', p') \mapsto (q, p) \in \mathcal{V}$ *a real analytic canonical transformation from a nbhd of the origin to a nbhd of the origin which puts* H *in normal form up to order* M *i.e.*

$$H \circ \tau = H_0 + Z + R$$

with

1. Z *is a polynomial of order* r *and is in normal form*
2. $R \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ *and* $R(z, \bar{z}) = O(\|(q, p)\|^{M+1})$
3. τ *is close to the identity:* $\tau(q, p) = (q, p) + O(\|(q, p)\|^2)$

Birkhoff Normal Form Theorem in Finite Dimension

Corollary

Assume ω is nonresonant. For each $M \geq 3$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\|(q_0, p_0)\| = \varepsilon < \varepsilon_0$ the solution $(q(t), p(t))$ of the Hamiltonian system associated to H which takes value (q_0, p_0) at $t = 0$ satisfies

$$\|(q(t), p(t)\| \leq 2\varepsilon \text{ for } |t| \leq \frac{C}{\varepsilon^{M-1}}.$$

Normal Form: Formal Argument

Consider the ODE

$$i\partial_t x_n = \omega_n x_n + \sum_{k\geq 2} \left(f_k(x)\right)_n$$

With

- Auxiliary Hamiltonian: $\chi(x)$
- X_χ the corresponding vector field

We note that for any vector field Y, its transformed vector field under the time 1 flow generated by X_χ is

$$e^{\mathrm{ad}_{X_\chi}} Y = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}^k_{X_\chi} Y \tag{8}$$

where $\mathrm{ad}_X Y := [Y, X]$.

Iterative Step

- Let χ be degree $K_0 + 1$
- Let $\Phi_\chi(x)$ be the time-1 flow map associated with the Hamiltonian vector field X_χ.
- Consider the change of variables $y = \Phi_\chi(x)$
- Using the identity (8), one obtains

$$i\partial_t y_n = \omega_n y_n + \sum_{k=2}^{K_0-1} \left(f_k(y)\right)_n + ([X_\chi, \omega y](y))_n + (f_{K_0}(y))_n + h.o.t.$$

Plan: choose χ and another vector-valued homogeneous polynomial of degree K_0, R_{K_0}, in such a way that we can decompose f_{K_0} as follows

$$f_{K_0}(y) = R_{K_0}(y) - [X_\chi, \omega y](y) \tag{9}$$

- We can find χ so that R_{K_0} is in the kernel of the following function

$$\mathrm{ad}_\omega(X) := [X, \omega y].$$

- Any $Y \in \ker \mathrm{ad}_\omega$ is referred to as "normal" or "resonant".

Appearance of small divisors

- Condition for a monomial, $y^\alpha \bar{y}^\beta \partial_{y_m}$, $(\alpha, \beta \in \mathbb{N}^\infty)$ to satisfy $y^\alpha \bar{y}^\beta \partial_{y_m} \in \ker \mathrm{ad}_\omega$:

$$\mathrm{ad}_\omega(y^\alpha \bar{y}^\beta \partial_{y_m}) = [(\alpha - \beta) \cdot \omega - \omega_m] y^\alpha \bar{y}^\beta \partial_{y_m}$$

- For individual terms, (9) becomes

$$R_{\alpha,\beta,m} - (\omega \cdot (\alpha - \beta) - \omega_m) X_{\alpha,\beta,m} = f_{\alpha,\beta,m}$$

Appearance of small divisors

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- For individual terms, (9) becomes

$$R_{\alpha,\beta,m} - (\omega \cdot (\alpha - \beta) - \omega_m) X_{\alpha,\beta,m} = f_{\alpha,\beta,m}$$

- Definition of X_χ and R_{K_0}:

$$\begin{matrix} R_{\alpha,\beta,m} := f_{\alpha,\beta,m} \\ X_{\alpha,\beta,m} := 0 \end{matrix} \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m = 0$$

$$X_{\alpha,\beta,m} := \frac{-f_{\alpha,\beta,m}}{(\omega \cdot (\alpha - \beta) - \omega_m)} \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m \neq 0$$

Appearance of small divisors

- In finite dimension,

$$\inf\{|\omega\cdot(\alpha-\beta)-\omega_m| \,\big|\, \omega\cdot(\alpha-\beta)-\omega_m\neq 0\}>0$$

- Leads to bound on change-of-variables map(symplectomorphism).
- Not necessarily true in infinite dimensions.

Nonresonance Condition

Definition (Nonresonance Condition)

There exists $\gamma = \gamma_M > 0$ and $\tau = \tau_M > 0$ such that for any N large enough, one has

$$\left| \sum_{q \geq 1} \lambda_q \omega_q \right| \geq \frac{\gamma}{N^\tau} \qquad \text{for } \|\lambda\|_1 \leq M, \ \sum_{q > N} |\lambda_q| \leq 2 \qquad (10)$$

where $\lambda \in \mathbb{Z}^\infty \setminus \{0\}$.

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$$\left|\sum_{q\geq 1} \lambda_q \omega_q\right| \geq \frac{\gamma}{N^\tau} \qquad \text{for } \|\lambda\|_1 \leq M, \quad \sum_{q>N} |\lambda_q| \leq 2 \qquad (10)$$

where $\lambda \in \mathbb{Z}^\infty \setminus \{0\}$.

The following generalization of the "non-resonance" result in Bambusi-Grebert holds.

Theorem (Bambusi-Grebert 2006)

For any $L_0 > 0$, there exists a set $J \subset (0, L_0)$ of full measure such that if $L \in J$ then for any $M > 0$ the Nonresonance Condition holds.

Definition

For $x = \{x_n\}_{n \in \mathbb{Z}^d}$, define the standard Sobolev norm as

$$\|x\|_s := \sqrt{\sum_{n \in \mathbb{Z}^d} |x_n|^2 \langle n \rangle^{2s}}$$

Define H^s as

$$H^s := \{x = \{x_n\}_{n \in \mathbb{Z}^d} \mid \|x\|_s < \infty\}$$

Let

$$\tilde{X}(z) := \sum_{\|\alpha\|_{\ell_1}=\ell} |X_\alpha| z^\alpha$$

Definition (Tame Modulus)

Let X be a vector-valued homogeneous polynomial of degree ℓ. X is said to have s-tame modulus if there exists $C > 0$ such that

$$\left\|\tilde{X}(z^{(1)}, ..., z^{(\ell)})\right\|_s$$

$$\leq C\frac{1}{\ell}\sum_{k=1}^{\ell} \|z^{(1)}\|_{\frac{d+1}{2}} \cdots \|z^{(k-1)}\|_{\frac{d+1}{2}} \|z^{(k)}\|_s \|z^{(k+1)}\|_{\frac{d+1}{2}} \cdots \|z^{(\ell)}\|_{\frac{d+1}{2}}$$

for all $z^{(1)}, ..., z^{(\ell)} \in H^s$. The infimum over all C for which the inequality holds is called the tame s-norm and is denoted $|X|_s$.

Normal Form Theorem

Theorem (Bambusi-Grebert 2006)

Consider the equation

$$i\dot{x} = \omega x + \sum_{k \geq 2} f_k(x). \tag{11}$$

and assume the nonresonance condition (10). *For any* $M \in \mathbb{N}$, *there exists* $s_0 = s_0(M, \tau)$ *such that for any* $s \geq s_0$ *there exists* $r_s > 0$ *such that for* $r < r_s$, *there exists an analytic canonical change of variables*

$$y = \Phi^{(M)}(x)$$
$$\Phi^{(M)} : B_s(r) \to B_s(3r)$$

which puts (11) *into the normal form*

$$i\dot{y} = \omega y + \mathcal{R}^{(M)}(y) + \mathcal{X}^{(M)}(y). \tag{12}$$

Normal Form Theorem continued

Theorem (Theorem cont.)

Moreover there exists a constant $C = C_s$ such that:

- $$\sup_{x \in B_s(r)} \|x - \Phi^{(M)}(x)\|_s \leq Cr^2$$
- *$\mathcal{R}^{(M)}$ is at most of degree $M + 2$, is resonant, and has tame modulus*
- *the following bound holds*

$$\|\mathcal{X}^{(M)}\|_{s,r} \leq Cr^{M+\frac{3}{2}}$$

Normal Form Theorem Ideas

- In the homological equation

$$f_{K_0}(y) = R_{K_0}(y) - [X_\chi, \omega y](y)$$

let $f_{K_0} = \tilde{f} + f^*$

Normal Form Theorem Ideas

- In the homological equation

$$f_{K_0}(y) = R_{K_0}(y) - [X_\chi, \omega y](y)$$

let $f_{K_0} = \tilde{f} + f^*$

- f^* consists of terms, $f_{\alpha,\beta,m} y^\alpha \bar{y}^\beta \partial_{y_m}$ where

$$\sum_{|n_i|>N} |\alpha_{n_i}| + \sum_{|m_i|>N} |\beta_{m_i}| + \mathbb{1}_{\{|n|>N\}}(m) \leq 2$$

- $\tilde{f}$ is small when $\|y\|_s$ is small due to Tame Modulus.
- We instead solve

$$f^*(y) = R_{K_0}(y) - [X_\chi, \omega y](y)$$

Main Theorem: Statement from FGL '13

Theorem (Faou, Gauckler, Lubich 2013)

Let $\rho_0 > 0$ be such that $1 - 2\lambda\rho_0^2 > 0$, and let $M > 1$ be fixed arbitrarily. There exists $s_0 > 0$, $C \geq 1$ and a set of full measure $\mathcal{P}$ in the interval $(0, \rho_0]$ such that for every $s \geq s_0$ and every $\rho \in \mathcal{P}$, there exists ε_0 such that for every $m \in \mathbb{Z}^d$ the following holds: if the initial data $u(\bullet, 0)$ are such that

$$\|u(\bullet,0)\|_{L^2} = \rho \qquad \text{and} \qquad \|e^{-im\cdot\bullet}u(\bullet,0) - u_m(0)\|_{H^s} = \varepsilon \leq \varepsilon_0$$

then the solution of (1) (with $p = 1$) with these initial data satisfies

$$\|e^{-im\cdot\bullet}u(\bullet,t) - u_m(t)\|_{H^s} \leq C\varepsilon \text{ for } t \leq \varepsilon^{-M}$$

Structure of the cubic case

Let

$$H_c = \int_{\mathbb{T}} (|\partial_x u|^2 + |u|^4)\, dx$$

Theorem (Kappeler, Grebert 2014)

There exists a bi-analytic diffeomorphism $\Omega : H^1 \to H^1$ such that Ω introduces Birkhoff coordinates for NLS on H^1. That is, on H^1 the transformed NLS Hamiltonian $H_c \circ \Omega^{-1}$ is a real-analytic function of the actions

$$I_n = \frac{|x_n|^2}{2}$$

for $n \in \mathbb{Z}$. Furthermore, $d_0\Omega$ is the Fourier transform.

Main Theorem: Statement

Theorem (W. 2014)

Let $L_0 > 0$ be such that $1 - 2p\lambda L_0^p > 0$, and let $M > 1$ be fixed arbitrarily. There exists $s_0 > 0$, $C \geq 1$ and a set of full measure $\mathcal{P}$ in the interval $(0, L_0]$ such that for every $s \geq s_0$ and every $L \in \mathcal{P}$, there exists ε_0 such that for every $m \in \mathbb{Z}^d$ the following holds: if the initial data $u(\bullet, 0)$ are such that

$$\|u(\bullet,0)\|_{L^2}^2 = L \qquad \text{and} \qquad \|e^{-im\cdot\bullet}u(\bullet,0) - u_m(0)\|_{H^s} = \varepsilon \leq \varepsilon_0$$

then the solution of (1) with these initial data satisfies

$$\|e^{-im\cdot\bullet}u(\bullet,t) - u_m(t)\|_{H^s} \leq C\varepsilon \text{ for } t \leq \varepsilon^{-M}$$

Characterization of $\mathcal{R}^{(M)}$

Proposition

The truncation of (12),

$$i\dot{y} = \omega y + \mathcal{R}^{(M)}(y)$$

can be decoupled in the following way:

$$i\partial_t \begin{pmatrix} y_{n_1} \\ \cdots \\ y_{n_k} \end{pmatrix} = \mathcal{M}_q \begin{pmatrix} y_{n_1} \\ \cdots \\ y_{n_k} \end{pmatrix} \tag{13}$$

where $q \geq 1$, $\{n_1, \ldots, n_k\} := \{n \in \mathbb{Z}^d \ : \ |n| = q\}$,
$\mathcal{M}_q = \mathcal{M}_q(\omega, \{y_j\})$ *is a self-adjoint matrix for all* t.

Characterization of $\mathcal{R}^{(M)}$

The form of the resonant terms depends entirely on two properties of the Hamiltonian:

- The Hamiltonian obeys the Conservation of Momentum law:
  - For any monomial, $f_{\alpha,\beta,m} y^{\alpha} \bar{y}^{\beta} \partial_{y_m}$, in the vector field, the indices satisfy

$$\sum \alpha_k k - \sum \beta_j j - m = 0$$

- $\{\omega_q\}_{q<N}$ is a linearly independent set

Resonant Dynamics

Proposition

Suppose $y \in H^s$ *satisfies* (13), *then*

$$\partial_t \|y\|_s^2 = \partial_t \sum_{q \geq 1} \left(\sum_{|n_i| = q} |y_{n_i}|^2 \right) \langle q \rangle^{2s} = 0$$

Further Questions

- Infinite time result?
- Feasibility of the Floquet/Duhamel iteration
- KAM result

KAM Result

- Obstacles
  - One parameter family of frequencies
  - Repeated frequencies
- May be able to overcome this: Bambusi, Berti, Magistrelli *Degenerate KAM theory for PDEs*

References

- D. Bambusi, B. Grébert, *Birkhoff normal form for partial differential equations with tame modulus*, Duke Math. J. **135** (2006), no. 3, 507–567.
- E. Faou, L. Gauckler, C. Lubich, *Sobolev stability of plane wave solutions to the cubic nonlinear Schrödinger equation on a torus*, Comm. Partial Differential Equations **38** (2013), no. 7, 1123–1140.
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Thank you for listening