The Frequency Bound of Donnelly-Fefferman

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In Donnelly - Fefferman, they prove Yau's conjecture for analytic manifolds.

The main result of their paper is an estimate for the order of vanishing of eigenfunctions.

Thm:
$$\frac{\sup_{B(x,2r)} |u_\lambda|}{\sup_{B(x,r)} |u_\lambda|} \lesssim 2^{c\sqrt{\lambda}}$$

Just like the almost monotonicity estimate of Garafalo - Lin, the proof is quite technical. We will follow the proof given in Mangoubi "The effect of curvature on convexity properties of harmonic functions and eigenfunctions".

Not only is it easier to read, Mangoubi proves an almost monotonicity result on the way to recreating the result of Donnelly - Fefferman.

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Donnelly - Fefferman.

The first to use this two step procedure to prove the doubling bound was Lin in "Nodal Sets in Solutions of Elliptic and Parabolic Equations".

<u>Thm</u> (Cor. 2.3 from Mangoubi)

Let N be a complete Riem. Man. of dimension $d+1$ w/ $|\mathrm{Sec}_N| \leq K$, Then

$$\frac{\int_{\partial B(2r)} |h|^2}{\int_{\partial B(r)} |h|^2} \leq \left(\frac{\int_{\partial B(2s)} |h|^2}{\int_{\partial B(s)} |h|^2} \right)^{1+r^2K}.$$

for $r < s < \frac{1}{4\sqrt{(d+1)K+1}}$

Using this almost-monotonicity estimate with the following comparison, we can show the frequency bound.

<u>Lem</u>: (Lem 3.1 Mangoubi). Let M be a compact Riem. Man. Fix $0 < \alpha < 1$. Then, for all $0 < r < R_0(M)$,

Fix $0<\alpha<1$. Then, for all $0<r<R_0(M)$,

$$C_{\alpha,M,\lambda} \sup_{B(\alpha r)} |u_\lambda|^2 \le \frac{\int_{\partial B(r)} |u|^2}{r^d} \le C_M \, e^{2r\lambda^{1/2}} \sup_{B(r)} |u_\lambda|^2.$$

From these two results, one obtains,

Thm (Thm 3.2)

$$\frac{\sup_{B(3r)} |u_\lambda|}{\sup_{B(2r)} |u_\lambda|} \lesssim e^{c_0 s \lambda^{1/2}} \left(\frac{\sup_{B(8s)} |u_\lambda|}{\sup_{B(3s)} |u_\lambda|} \right)^{1 + c_1 r^2 k}.$$

Fixing s gives us the result we desire.

It is clear that the Lemma follows from Hölders inequality for the upper bound and elliptic estimates for the lower bound

Thm (Cor 3.2) follows from showing

thm (or s.t.)

that $\log \int_{\partial B(r)} |h|^2$ satisfies a

2^{nd} order differential inequality

$$\left(\frac{d}{dr}\right)^2 \left(\log \int_{\partial B(r)} |h|^2\right) + C_0(M) \frac{d}{dr}\left(\log \int_{\partial B(r)} |h|^2\right) \geq -C_1(M)$$

for $C_1(M) > 0$.

which would be similar to the test for the log convexity of $\log \int_{\partial B(r)} |h|^2$ w.r.t. $\frac{-1}{r^{n-1}}$

$$\left(\frac{d}{dr}\right)^2 \left(\log \int_{\partial B(r)} |h|^2\right) + \frac{n-1}{r} \frac{d}{dr} \log\left(\int_{\partial B(r)} |h|^2\right) \geq 0.$$