In Donnelly - Fefferman, they prove Yau's conjecture for analytic manifolds. The main result of their paper is an estimate for the order of vanishing of eigenfunctions.

Thus:

$$\sup_{B(x, r)} |u| \leq 2e^{\sqrt{r}}$$

Just like the almost monotonicity estimate by GaraPalo - Lin, the proof is quite technical. We will follow the proof given in Mangoubi "The effect of curvature on convexity properties of harmonic functions and eigenfunctions". Not only is it easier to read, Mangoubi proves an almost monotonicity result on the way to recreating the result of Donnelly - Fefferman.
Donnelly - Fefferman.

The first to use this two step procedure to prove the doubling bound was Lin in "Nodal Sets in Solutions of Elliptic and Parabolic Equations."

Theorem (Con. 2.3 from Mangoubi)

Let $N$ be a complete Riem. Man. of dimension $d+1$ of $\text{sec} n \leq K$, Then

$$\frac{\int_{B(2r)} |h|^2}{\int_{B(r)} |h|^2} \leq \left( \frac{\int_{B(2r)} |h|^2}{\int_{B(r)} |h|^2} \right)^{1 + r^2 K}$$

for $\forall s \leq \frac{1}{4rK}$

Using this almost monotonicity estimate with the following comparison, we can show the frequency bound.

Lemma (Con. 3.1 Mangoubi).

Let $M$ be a compact Riem. Man.

Fix $0 < a < 1$. Then, for all $0 < r < R_0(M)$,
Fix \( 0 < \alpha < 1 \). Then, for all \( 0 < r \leq R_0(M) \),

\[
\sup_{x \in B(r)} \| u_x \|^2 \leq \frac{S h \| u \|^2}{r^d} \leq C \alpha e^{2 \pi \alpha \frac{1}{2} \sup_{B(1)} \| u_x \|^2}.
\]

From these two results, one obtains

**Theorem (Thm 3.2)**

\[
\sup_{B(3r)} \| u_x \|^2 \leq e^{c_5 \alpha^{1/2}} \left( \frac{\sup_{B(3r)} \| u_x \|^2}{\sup_{B(9r)} \| u_x \|^2} \right)^{1 + C_4 r^2}.
\]

Fixing \( s \) gives us the result we desire.

It is clear that the Lemma follows from Hölder's inequality for the upper bound and elliptic estimates for the lower bound.

**Theorem (Cor 3.2)** follows from showing
that \( \log S_1 \mu_1 \) satisfies a 2nd order differential inequality

\[
\left( \frac{d}{dr} \right)^2 \left( \log S_{\beta(r)} \mu_1 \right) + C_0(M) \frac{d}{dr} \left( \log S_{\beta(r)} \mu_1 \right) \geq -C_2(M)
\]

for \( C_2(M) > 0 \).

which would be similar to

the test for the log convexity

of \( \log S_{\beta(r)} \mu_1 \) w.r.t. \( \frac{1}{\rho - 1} \)

\[
\left( \frac{d}{dr} \right)^2 \left( \log S_{\beta(r)} \mu_1 \right) + \frac{n-1}{\rho - 1} \frac{d}{dr} \log S_{\beta(r)} \mu_1 \geq 0.
\]