Meta Argument of Logunov-Malinnikova

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The meta argument can be summarized in the following way

Fix λ, M, u satisfying

$$\Delta_M u + \lambda u = 0.$$

Let

$$N(u, Q) := \log \frac{\int_{\ell Q} |u|^2}{\int_Q |u|^2}$$

for fixed $\ell > 2\sqrt{d}$.

and $E_\lambda = \{u = 0\}$.

We consider E_λ at the $\lambda^{-1/2}$ length scale

$$\mathcal{H}^{d-1}(E_\lambda)$$

$$\|$$

$$\sum_{i=1}^{C\lambda^{d/2}} \mathcal{H}^{d-1}\left(E_\lambda \cap B(x_i, \lambda^{-1/2})\right)$$

$$\lesssim$$

$$\lambda^{d/2} \mathcal{H}^{d-1}\left(E_\lambda \cap B_{\lambda^{-1/2}}\right)$$

$$N(u, B_{\lambda^{-1/2}})^{-\alpha_1} \lesssim \frac{\mathcal{H}^{d-1}(E_\lambda \cap B_{\lambda^{-1/2}})}{\lambda^{-(d-1)/2}} \lesssim N(u, B_{\lambda^{-1/2}})^{\alpha_2}.$$

This not good enough, so we consider

$$\sum \frac{N(u, B_{r_k})^{-\alpha_1} r_k^{-(d-1)}}{\lambda^{-(d-1)/2}} \lesssim \sum \frac{\mathcal{H}^{d-1}(E_\lambda \cap B_{r_k})}{\lambda^{-(d-1)/2}} \lesssim \frac{\mathcal{H}^{d-1}(E_\lambda \cap B_{\lambda^{-1/2}})}{\lambda^{-(d-1)/2}}$$

Many balls with relatively small frequency.

$\boxed{\text{the argument of "Two and Three dimensions"}}$

In the paper, "Two and Three dimensions", Logunov and Malinnikova prove that in dimension 2,

$$\mathcal{H}^1(E_\lambda) \leq \lambda^{3/4 - \beta} \quad \text{for some } \tfrac{1}{4} > \beta > 0$$

and in dimension 3,

$$\mathcal{H}^2(E_\lambda) \geq \lambda^{\alpha} \quad \text{for some } \alpha > 0.$$

Logunov, of course, improves upon the lower bound in the last paper and provides upper bounds for higher dimensions in the second paper. However, the arguments in this paper are short and give a blueprint for the other arguments while demonstrating the possible obstacles.

1.) The local lower bound

Lemma: Let $u_\lambda(0)=0$, let $r \gtrsim \lambda^{-1/2}$

Suppose
$$\frac{\sup_{B_{r/2}} |u_\lambda|}{\sup_{B_{r/4}} |u_\lambda|} \leq 2^N \quad , N \geq 4 .$$

Then
$$\frac{\mathcal{H}^{d-1}(B_{r/2} \cap E_\lambda)}{r^{d-1}} \gtrsim N^{2-d}$$

Pf: From Harnack
$$\sup_{B_{r/4}} |u| \lesssim \sup_{\partial B_{\frac{3r}{8}}} |u|$$

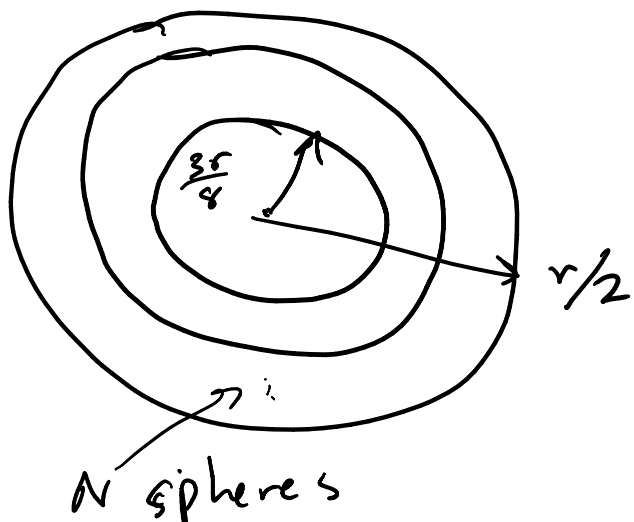
thus
$$\frac{\sup_{\partial B_r} \pm u}{\sup u} < \frac{\sup_{B_{r/2}} |u|}{\quad} < 2^N$$

Thus

$$\frac{\sup_{\partial B_r} u}{\sup_{\partial B_{\frac{3r}{4}}} u} = \frac{\sup_{\partial B_r} u}{\sup_{\partial B_{\frac{3r}{4}}} |u|} \lesssim \frac{\sup_{B_{r/2}} |u|}{\sup_{B_{\frac{r}{4}}} |u|} \leq 2^N$$

Since $u(x) = 0$, the mean value property of $h = e^{\lambda^{1/2} t} u$ implies that for every $r \lesssim \lambda^{-1/2}$ h takes positive and negative values inside.
Furthermore $\operatorname{sign} h = \operatorname{sign} u$.

Therefore, consider the concentric spheres



$$S_j = \left\{ |x| = r_j = r\left(\frac{3}{8} + \frac{j}{8N}\right) \right\} \quad j = 0, \dots, N$$

The stated mean value property implies

$$\sup_{S_j} u > 0 \quad \text{and} \quad \inf_{S_j} u < 0$$

Furthermore, since $\sup_{B_{r/2}} u > 0$ and $\inf u < 0$.

Furthermore, since $B^1_{r/2}$ $\inf_{B_{r/2}} u < 0$.

$$\sup_{S_j} u \le 2 \sup_{S_{j+1}} u$$

$$\left|\inf_{S_j} u\right| \le 2\left|\inf_{S_{j+1}} u\right|.$$

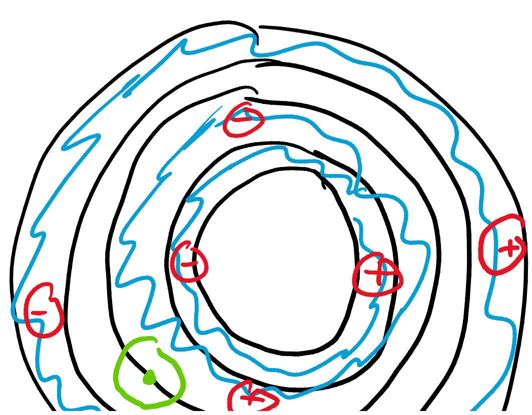
$$\Rightarrow \quad \frac{1}{2} \le \frac{\sup_{S_{j+1}} u}{\sup_{S_j} u}, \quad \frac{1}{2} \le \frac{\left|\inf_{S_{j+1}} u\right|}{\left|\inf_{S_j} u\right|}$$

Now we note that

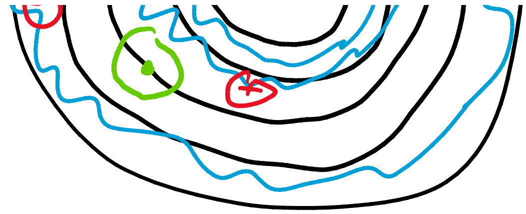
$$\frac{\sup_{S_1} u}{\sup_{B_{3r/8}} u} \cdot \;.\;.\; \cdots \frac{\sup_{B_{r/2}} u}{\sup_{S_{N-1}} u} = \frac{\sup_{B_{r/2}} u}{\sup_{B_{3r/8}} u} \lesssim 2^N.$$

$\Rightarrow$ at most $\frac{N}{4}$ ratios are larger than a constant.

$$\Rightarrow \frac{\sup_{S_{j+1}} u}{\sup_{S_j} u} \le C \quad \text{for} \sim N \quad j\text{'s}$$



~ good Annuli



Let $x_0 \in S_j$ be chosen so that

$$u(x_0) = \sup_{S_j} u.$$

Then $\sup_{B(x_0, \frac{r}{100N})} u \leq \sup_{B(x, r_{j+1})} u$

$$\lesssim \sup_{\partial B(x, r_{j+1})} u \lesssim \sup_{S_j} u$$

By maximum principle.

$$\sup_{B(x_0, \frac{r}{200N})} |u| \lesssim \sup_{S_j} u = u(x_0).$$

By gradient estimate, $\sup_{B(x_0, \frac{r}{1000N})} |\nabla u| \lesssim \frac{u(x_0)}{r}$.

$$\Rightarrow \inf_{B(x_0, c\frac{r}{N})} u > 0 \text{ for some } c.$$

$\Rightarrow$ in each good annulus, we can find a positive ball and negative ball, so

$$\mathcal{H}^{d-1}(E_\lambda \cap (B_{r_{k+1}} \setminus B_{r_{k-1}})) \gtrsim \left(\frac{r}{N}\right)^{d-1}.$$

holds for $\sim N$ annuli

$$\Rightarrow \mathcal{H}^{d-1}(E_\lambda \cap B_r) \gtrsim r^{d-1} N^{2-d}.$$

$\square$

$\checkmark \square$

We note that for $d=2$,

$$\frac{\mathcal{H}^{d-1}(B_{r/2}\cap E_\lambda)}{r^{d-1}} \geq N^{2-d} = C.$$

Would give Yau's lower bound which was proven by Brüning.

We proceed with a lower bound in dimension 3.

Lower bound in dimension 3

We use our standard trick of considering $h(x,t) = e^{\lambda^{1/2} t} u_\lambda(x)$ on $M \times I$.

Note: $\bar{N}(u_\lambda, Q) \sim \bar{N}(h, \tilde{Q})$

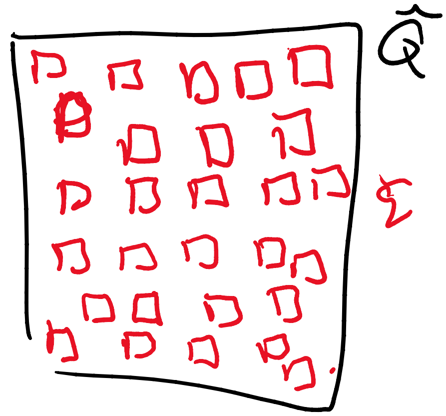
where $\tilde{Q} = Q \times J$ where $\operatorname{diam} J = \operatorname{diam} Q$.

By Donnelly-Fefferman $\bar{N}(h, \tilde{Q}) \lesssim \sqrt{\lambda}$

By Donnelly-Fefferman $N(u, Q) \lesssim \cdots$

for $\operatorname{diam} \tilde{Q} \lesssim \frac{1}{M}$.

Partition $\tilde{Q}$ into cubes, $\tilde{q}$, of diameter $\lambda^{-1/2}$ s.t. the center of each $\tilde{q}$ has a zero and so that we have $\sim \lambda^2$ $\tilde{q}$'s



By combinatorial lemma, half of all $\tilde{q}$ have

$$\bar{N}(u, \tilde{q}) \lesssim \frac{\lambda^{1/2}}{(\lambda^2)^\delta}$$

$\Rightarrow$ half of $\lambda^{-1/2}$ length cubes in Q satisfy $\bar{N}(u_\lambda, q) \lesssim \lambda^{1/2 - 2\delta}$

Now we just plug in

$$\frac{\mathcal{H}^2(B_{\lambda^{-1/2}} \cap E_\lambda)}{\lambda^{-1}} \gtrsim \lambda^{-1/2 + 2\delta}$$

$$\Rightarrow \mathcal{H}^2(B_{\lambda^{-1/2}} \cap E_\lambda) \gtrsim \lambda^{-3/2 + 2\delta}$$

$$\rightarrow \mathcal{H}^2(E_\lambda) \gtrsim \lambda^{2\delta} \quad \square$$

... in dimension 2

The upper bound in dimension 2

The benefit of considering a surface is that locally we can represent M in isothermal coordinates: $g_{ij} = \begin{pmatrix} \psi & 0 \\ 0 & \psi \end{pmatrix}$.

so

$$\Delta_M u_\lambda + \lambda u = 0$$

becomes in local coordinates

$$\Delta_{\mathbb{R}^2} u_\lambda + \frac{r^2 \lambda}{\psi^2} f = 0$$

From Donnelly - Fefferman 1990,

$$g : Q \to \mathbb{R}, \quad \overline{N}(g, Q) \le \Gamma, \quad \Delta g = \Gamma \psi g,$$

where $\|\psi\|_\infty =$ small, then

$$\mathcal{H}^1\left(\tfrac{1}{100} Q \cap \{g = 0\}\right) \lesssim \Gamma$$

apply at scale $r \sim \lambda^{-1/4}$

then

$$\frac{\mathcal{H}^1\left(\frac{1}{100} Q \cap \{g=0\}\right)}{\lambda^{-1/4}} \le \lambda^{1/2}$$

$$\Rightarrow \mathcal{H}^1\left(\tfrac{1}{100} Q \cap \{g=0\}\right) \lesssim \lambda^{1/4}$$

Since the are $\lambda^{1/2}$ such cubes,

3/4

Since the are $\lambda^{\cdot\cdot}$ such .

we have $\mathcal{H}^1(\{u_\lambda = 0\}) \lesssim \lambda^{3/4}$.

In Logunov - Malinnikova they are able to conclude that there are many small cubes in cubes of side-length $\lambda^{-1/4}$ with relatively small frequency. We then gain

$$\mathcal{H}^1(\{u_\lambda = 0\}) \lesssim \lambda^{3/4(1-\eta)}.$$

for some $\eta > 0$.