The meta argument can be summarized in the following way.

Fix \( \lambda, M, u \) satisfying
\[
\Delta_M u + \lambda u = 0.
\]

Let
\[
N(u, Q) = \log \frac{\int_Q |u|^{1/2}}{\int_Q |u|^{1/2}}
\]
for fixed \( \lambda > 2\sqrt{d} \).

and \( E_{\lambda} = \mathbb{E}_u = 0 \).

We consider \( E_{\lambda} \) at the \( \lambda^{-1/2} \) length scale

\[
\mathcal{N}_{\lambda}(E_{\lambda}) \leq \sum_{i=1}^{C \lambda^{d/2}} \mathcal{N}_{\lambda^{-1}}(E_{\lambda} \cap B(x_i, \lambda^{-1/2})) \leq \lambda^{d/2} \mathcal{N}_{\lambda^{-1}}(E_{\lambda} \cap B_{\lambda^{-1/2}})
\]
\[ N(u, B_{x/2}) \leq \mathcal{N}^{-1}\left( E \cap B_{x/2} \right) \leq N(u, B_{x/2}) \]

This is not good enough, so we consider

\[ \sum_{x} \frac{N(u, B_{x/2})}{x} \leq \frac{\mathcal{N}^{-1}(E \cap B_{x/2})}{x} \]

Many balls with relatively small frequency.

The argument of "Two and Three dimensions"

In the paper, "Two and Three dimensions", Logunov and Malinnikova prove that in dimension 2,

\[ \mathcal{N}^1(E_x) \leq x^{\frac{3}{4} - \beta} \text{ for some } \beta > 0 \]

and in dimension 3,

\[ \mathcal{N}^2(E_x) \geq x^a \text{ for some } a > 0. \]
Let $\nu(0) = 0$, let $r = \lambda^{-\frac{1}{2}}$

Suppose $\sup_{B_{r/2}} |u| \leq 2^N$, $N \geq 4$

Then $\mathcal{H}^{d-1}(B_{r/2} \cap E_x) \geq N^{2-d}$

**Proof:** From Harnack

$$\sup_{B_{r/2}} |u| \leq \sup_{\partial B_{r/2}} |u|$$

Thus $\sup_{\partial B_{r/2}} |u| < \sup_{B_{r/2}} |u| < 2^N$
\[ \sup_{\overline{B_r}} u = \frac{\sup_{\overline{B_r}} |u|}{\sup_{\overline{B_{r/2}}} |u|} \leq \frac{\sup_{B_r} \frac{1}{u}}{\sup_{B_{r/2}} \frac{1}{u}} = 2^N \]

Since \( u(x) = 0 \) is the mean value property at \( h = e^{x^2} \), it implies that for every \( r \leq 1^{-1/2} \), \( h \) takes positive and negative values inside. Furthermore, \( \text{sign } h = \text{sign } u \).

Therefore, consider the concentric spheres

\[ S_j = \left\{ |x| = r_j = r \left( \frac{3}{8} + \frac{j}{8N} \right) \right\} \quad j = 0, \ldots, N \]

The stated mean value property implies \( \sup_{S_j} u > 0 \) and \( \inf_{S_j} u < 0 \).

Furthermore, since \( \sup_{B_{r/2}} u > 0 \) and \( \inf_{B_{r/2}} u < 0 \).
Furthermore, since \( B_{1/2} \cap B_{3/2} = \emptyset \),

\[
\sup_{s_j \in s_{j+1}} u \leq 2 \sup_{s_j} u
\]

\[
\inf_{s_j} u \leq 2 \inf_{s_{j+1}} u
\]

\[
\Rightarrow \quad \frac{1}{2} \leq \frac{\sup_{s_j} u}{\sup_{s_{j+1}} u}
\]

\[
\leq \frac{1}{2} \leq \frac{\inf_{s_j} u}{\inf_{s_{j+1}} u}
\]

Now we note that

\[
\sup_{s_{j+1}} u = \sup_{s_{N-1}} u = \sup_{s_{N}} u \leq 2^N.
\]

Thus at most \( N/4 \) ratios are larger than a constant.

\[
\Rightarrow \quad \frac{\sup_{s_{j+1}} u}{\sup_{s_j} u} \leq C
\]

for uN j's.
Let \( x_0 \in S_j \) be chosen so that

\[ u(x_0) = \sup_{S_j} u. \]

Then

\[ \sup_{B(x_0, r \alpha(\log N)^{\frac{1}{d}})} u \leq \sup_{S_j} u \]

\[ \leq \sup_{\partial B(x_0, r \alpha(\log N)^{\frac{1}{d}})} u \leq \sup_{S_j} u. \]

By maximum principle,

\[ \sup_{B(x_0, r \alpha(\log N)^{\frac{1}{d}})} |u| = \sup_{S_j} u = u(x_0). \]

By gradient estimate,

\[ \sup_{B(x_0, r \alpha(\log N)^{\frac{1}{d}})} \nabla u | \leq C \frac{1}{r}. \]

\[ \implies \inf_{B(x_0, r \alpha(\log N)^{\frac{1}{d}})} u > 0 \text{ for some } c. \]

\[ \implies \text{in each good annulus we can find a positive ball and negative ball, so} \]

\[ \mathcal{H}^{d-1}(E_x \cap (B_{r_{\text{ext}}}, B_{r_{\text{int}}})) \geq \left( \frac{c}{\alpha} \right)^{d-1}. \]

holds for \( \alpha \sim N \) annuli

\[ \implies \mathcal{H}^{d-1}(E_x \cap B_r) \geq r^{d-1} N^{2-d}. \]
We note that for $d=2$,

$$
\mathcal{H}^{d-1}(B_{\sqrt{2}} \cap \mathcal{E}) \geq N^{2d} / C.
$$

Would give Yau's lower bound which was proven by Grüning.

We proceed with a lower bound in dimension 3.

**Lower bound in dimension 3**

We use our standard trick of considering $h(x, t) = e^{\sqrt{2} t} u_x(x)$ on $M \times \mathbb{I}$.

Note: $\bar{N}(u_x, a) \leq \bar{N}(u, \tilde{a})$

where $\tilde{a} = a \times S$ where $\text{diam } S = \text{diam } a$.

By Donnelly-Fefferman $\bar{N}(u, \tilde{a}) \leq \sqrt{\lambda}$.
By Donnelly-Fefferman, $N(n, Q) = \ldots$
for $\dim Q \leq n$.

Partition $Q$ into cubes, $\bar{Q}$, of diameter $\lambda^{-1/2}$ s.t. the center of each $\bar{Q}$ has a zero and so that we have $n \geq 2$ $\bar{Q}$'s.

By combinatorial lemma, half of all $\bar{Q}$ have

$$N(n, \bar{Q}) \leq \frac{\lambda^{1/2}}{\gamma}$$

$$\Rightarrow \text{half of } \lambda^{1/2} \text{ length cubes in } Q$$

satisfy $N(n, Q) \leq \lambda^{3/2 - 2}\gamma$.

Now we just plug in

$$\mathcal{H}^2(\cap_{x \in X_n} \mathcal{E}_x) \geq \lambda^{-1}$$

$$\Rightarrow \mathcal{H}^2(\cap_{x \in X_n} \mathcal{E}_x) \geq 2^{-1} \lambda^{-3/2 + 2\gamma}$$

$$\Rightarrow \mathcal{H}^2(\mathcal{E}_x) \geq 2^{2\gamma} \Box$$

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... in dimension 2
The upper bound in dimension 2

The benefit of considering a surface is that locally we can represent \( M \) in isothermal coordinates: \( g_{ij} = (\phi^0, \phi^4) \).

\[
\Delta \mu + \lambda u = 0
\]

becomes in local coordinates

\[
\Delta_{\text{loc}} u_{ij} + \frac{\partial g_{ij}}{\partial x^k} f = 0
\]

From Donnelly-Fefferman 1990,

\[
g : \mathbb{R} \to \Gamma, \quad \bar{N}(g, \phi) \leq \Gamma, \quad \Delta g = \Gamma^{-2} \psi_g
\]

where \( \| \psi_g \|_\infty \) is small, then

\[
\mathcal{H}^1\left( \frac{1}{100} \mathcal{B} \cap \bar{g} = 0^3 \right) \leq \Gamma
\]

apply at scale \( r \sim \lambda^{-\frac{1}{4}} \)

then \( \mathcal{H}^1\left( \frac{1}{100} \mathcal{B} \cap \bar{g} = 0^3 \right) \leq \lambda^{\frac{1}{2}} \)

\[
\lambda^{-\frac{1}{4}} \lambda^{\frac{1}{2}} = \lambda^{\frac{1}{4}}
\]

\[
\Rightarrow \quad \mathcal{H}^1\left( \frac{1}{100} \mathcal{B} \cap \bar{g} = 0^3 \right) \leq \lambda^{\frac{1}{4}}
\]

Since there are \( \lambda^{\frac{1}{2}} \) such cubes,
Since the are $x$ such we have $N'(2x^3)$ $\leq x^{3/4}$.

In Lagunov–Malinnikov they are able to conclude that there are many small cubes in cubes of side-length $x^{1/4}$ with relatively small frequency. We then again

$$N'(2y^3) \leq x^{3/4}(1-y).$$

For some $y > 0$. 