LOGARITHMIC CONVEXITY FOR SUPREMUM NORMS OF HARMONIC FUNCTIONS

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ABSTRACT

We prove the following convexity property for supremum norms of harmonic functions. Let Ω be a domain in $\mathbf{R}^n$, Ω_0 and E a subdomain and a compact subset of Ω, respectively. Then there exists a constant $\alpha = \alpha(E, \Omega_0, \Omega) \in (0, 1]$ such that for all harmonic functions u on Ω, the inequality

$$\|u\|_E \leqslant \|u\|_{\Omega_0}^{\alpha} \|u\|_{\Omega}^{1-\alpha}$$

is valid. The case of concentric balls $\Omega_0 \subset E \subset \Omega$ plays a key role in the proof. For positive harmonic functions on such balls, we determine the sharp constant α in the inequality.

1. *Introduction and results*

If an analytic function f on a domain Ω in $\mathbf{C}$ is bounded by 1 on Ω, while it is 'exponentially small' on a subdomain Ω_0 or its boundary,

$$|f| \leqslant e^{-K} \quad \text{on } \partial\Omega_0,$$

for some positive number K, then f is 'exponentially small' everywhere in Ω. More precisely, for $z \in \Omega \backslash \overline{\Omega_0}$, there is a positive constant $\alpha = \alpha(z)$ independent of f and K such that

$$|f(z)| \leqslant e^{-\alpha K}. \tag{1.1}$$

It may seem surprising that there is similar 'propagation of smallness' for arbitrary harmonic functions in $\mathbf{R}^n$ ($n \geqslant 2$).

THEOREM 1.1. *Let Ω be a domain in $\mathbf{R}^n$, $\Omega_0 \subset \Omega$ a (nonempty) subdomain and $E \subset \Omega$ a (nonempty) compact subset. Then there is a constant $\alpha = \alpha(E, \Omega_0, \Omega) \in (0, 1]$ such that for all complex-valued harmonic functions u on Ω,*

$$\|u\|_E \leqslant \|u\|_{\Omega_0}^{\alpha} \|u\|_{\Omega}^{1-\alpha}, \tag{1.2}$$

where $\|u\|_A = \sup_{x \in A} |u(x)|$.

For analytic functions f in $\mathbf{C}$, (1.1) follows immediately from Nevanlinna's two-constants theorem. In this case, $\log|f|$ is subharmonic and one may take $\alpha(z)$ equal to the harmonic measure $\omega(z)$ of the boundary of Ω_0 relative to $\Omega \backslash \overline{\Omega_0}$ [8]. Observe that the proof requires analyticity of f only on $\Omega \backslash \overline{\Omega_0}$. Hadamard's three-circles theorem may be considered as an important special case.

For analytic functions f in $\mathbf{C}^n$ one may also work with $\log|f|$. However, in the general harmonic case no special property of $\log|u|$ is known to yield (1.2). In fact,

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for real harmonic u, straightforward use of harmonic measure gives only an arithmetic inequality which is much weaker than (1.2) when $|u|$ is very small on Ω_0; compare Lemma 2.2.

For our proof of Theorem 1.1, we start with the special case of concentric balls (Section 3). Subsequently, the general case is dealt with by a covering argument of known type (Section 4).

THEOREM 1.2 (Three-balls theorem). *Suppose $0<\rho<r<R$ and $n\geqslant 2$. Then there exists a constant $\alpha\in(0,1)$, depending only on ρ/R, r/R and n, such that for all complex-valued harmonic functions u on the ball $B(0,R)$ in $\mathbf{R}^n$,*

$$\|u\|_r \leqslant \|u\|_\rho^\alpha \, \|u\|_R^{1-\alpha}, \tag{1.3}$$

where $\|u\|_t=\sup|u(x)|$ on the ball $B(0,t)$.

For the special case of *positive* harmonic functions on balls, Section 5 contains an explicit description of the optimal constant α. That work has benefited greatly from a visit to Amsterdam of Yu. I. Lyubarskii (Kharkov).

For arbitrary harmonic functions we have obtained a partial solution of the relevant extremal problem (Section 5).

We finally mention some earlier work on 'transfer of smallness'. Naturally, an inequality (1.3) (with sup norms over spheres) cannot hold for all harmonic functions u on the spherical shell bounded by the spheres $S(0,\rho)$ and $S(0,R)$, since such functions may vanish on the inner sphere without vanishing identically. However, there is a (fairly complicated) three-spheres theorem for harmonic functions by Solomentsev [**10**]. In this context one may also mention work by Janson and Peetre [**5**] and very recent work by Peetre and Sjölin [**9**]. A certain transfer of smallness for solutions of elliptic partial differential equations has been observed by Armitage, Bagby and Gauthier [**1**]. It is plausible that our Theorem 1.1 can be extended to solutions of a large class of partial differential equations. (In fact, stimulated by our work, R. G. M. Brummelhuis has just obtained a crucial L^2 analogue of Theorem 1.2 for such equations.) A preliminary form of the present results has appeared in the second author's PhD thesis [**6**].

2. *Auxiliary formulas*

It is useful to recall some basic properties of bounded harmonic functions u on the ball $B(0,R)$ in $\mathbf{R}^n$; compare [**4**]. Assuming that u is continuous on the closed ball $\bar{B}(0,R)$, one can represent $u(x)$ by the Poisson integral of its boundary values on the sphere $S(0,R)$:

$$u(x)=\frac{1}{\sigma_n}\int_{S(0,R)}\frac{R^2-|x|^2}{R|y-x|^n}u(y)\,d\sigma(y), \quad x\in B(0,R). \tag{2.1}$$

Here $\sigma_n=2\pi^{n/2}/\Gamma(\frac{1}{2}n)$ is the surface area of the unit sphere $S=S(0,1)$ in $\mathbf{R}^n$. Formula (2.1) holds for all bounded harmonic functions on the ball if one substitutes for $u(y)$ the (almost everywhere existing) nontangential boundary values of u. Indeed, for fixed x one can start with a smaller ball and then take a limit.

Setting $x=r\xi$ where $\xi\in S$, u may also be represented by its Laplace series

$$u(r\xi)\sim\sum_{k=0}^{\infty}Y_k(\xi)\,r^k, \quad 0\leqslant r\leqslant R. \tag{2.2}$$

Here the spherical harmonics Y_k are pairwise orthogonal in $L^2(S)$, and for fixed r, the series is convergent to $u(r\xi)$ in $L^2(S)$. By this representation one has the following identity for L^2-norms on spheres:

$$\|u\|_{r,2}^2 \overset{\text{def}}{=} \frac{1}{m(S(0,r))}\int_{S(0,r)} |u(x)|^2\, d\sigma(x)$$
$$= \frac{1}{\sigma_n}\int_{S(0,1)} |u(r\xi)|^2\, d\sigma(\xi) = \sum_{k=0}^{\infty} \|Y_k\|_2^2\, r^{2k}. \tag{2.3}$$

Hence, by Hadamard's three-circles theorem applied to $\sum \|Y_k\|^2 z^{2k}$ or by direct computation, $\log \|u\|_{r,2}$ is a convex function of $\log r$.

LEMMA 2.1. *For $0 < \rho < t < R$,*

$$\|u\|_{t,2} \leqslant \|u\|_{\rho,2}^{\beta}\, \|u\|_{R,2}^{1-\beta},$$

where β is Hadamard's exponent:

$$\beta = \beta_H\left(\frac{\rho}{R}, \frac{t}{R}\right) = \frac{\log t/R}{\log \rho/R}.$$

The maximum principle for harmonic functions on a spherical shell $B(0,R) - \bar{B}(0,\rho)$ gives a corresponding *arithmetic* inequality for supremum norms on spheres which in our case agree with the supremum norms $\|u\|_r$ over balls.

LEMMA 2.2. *For $0 < \rho < t < R$,*

$$\|u\|_t \leqslant \frac{t^{2-n} - R^{2-n}}{\rho^{2-n} - R^{2-n}}\|u\|_\rho + \frac{\rho^{2-n} - t^{2-n}}{\rho^{2-n} - R^{2-n}}\|u\|_R, \quad n \geqslant 3.$$

For $n = 2$, the right-hand side has to be replaced by its limit as $n \to 2$.

For bounded harmonic functions on the unit ball with axial symmetry about the x_1-axis, the values on this axis are given by a simple integral. It is obtained from the Poisson integral (2.1) by setting $R = 1$, $x = se_1$, $y = (y_1, y') = (\cos\theta, \sin\theta\cdot\eta')$, where θ runs over $[0,\pi]$ and η' runs over the unit sphere S' in $(x_2, \ldots, x_n)$-space. In this case $d\sigma(y) = (\sin\theta)^{n-2}\, d\theta\, d\sigma(\eta')$. Integration over S' gives the following.

LEMMA 2.3. *Let $u(x) = u(x_1, x')$ be harmonic and bounded on the unit ball $B(0,1)$, and let $u(x)$ depend only on x_1 and the length of $x' = (x_2, \ldots, x_n)$. Then the values $u(se_1) = u(s, 0')$ are given by the integral*

$$u(se_1) = \int_0^{\pi} Q(s,\theta)\, u(\cos\theta, \sin\theta\cdot e_2)\, d\theta, \quad -1 < s < 1, \tag{2.4}$$

where

$$Q(s,\theta) = \frac{\sigma_{n-1}}{\sigma_n}\frac{(1-s^2)\sin^{n-2}\theta}{(1+s^2-2s\cos\theta)^{n/2}}. \tag{2.5}$$

We also need a related special integral.

LEMMA 2.4. *For $n \geqslant 2$ and $a > b \geqslant 0$,*

$$\frac{\sigma_{n-1}}{\sigma_n}\int_0^\pi \frac{\sin^{n-2}\theta}{(a-b\cos\theta)^n}\,d\theta = \frac{a}{(a^2-b^2)^{(n+1)/2}}.$$

For the proof, one may start with the integral of $(a-b\cos\theta)^{-1}$ and carry out differentiations with respect to a and b. The case where the integrand has denominator $(a-b\cos\theta)^{n-1}$ occurs in the table in [3, p. 384].

3. *Proof of the three-balls theorem*

In the present section we do not aim for the best constant α in inequality (1.3).

Changing scale in Theorem 1.2, one may take $R = 1$. It may also be assumed that our harmonic function u on $B(0,1)$ is bounded and that $\|u\|_1 = 1$. (If $\|u\|_\rho = 0$ or, equivalently, $u \equiv 0$, there is nothing to prove.) We may, finally, impose the following conditions, using operations which do not increase the norms $\|u\|_s$ and which leave $\|u\|_r$ unchanged.

(i) For the given r, the norm $\|u\|_r$ is equal to the value of u at the point re_1. (One may rotate about the origin and multiply by a constant of absolute value 1.)

(ii) u is real-valued. (With (i) satisfied, u may be replaced by $\operatorname{Re} u$.)

(iii) u has axial symmetry about the x_1-axis. (With (i) satisfied, symmetrization of u with respect to the x_1-axis—which replaces u by an average of rotations of u—leads to a function $\tilde{u}$ with $\|\tilde{u}\|_r = u(re_1) = \tilde{u}(re_1)$.)

DEFINITION 3.1. For fixed $0 < \rho < r < 1$, $n \geqslant 2$ and $0 < \varepsilon \leqslant 1$, $H_\varepsilon = H(\varepsilon,\rho,r,n)$ will denote the class of those harmonic functions u on the unit ball in $\mathbf{R}^n$ that satisfy (i)–(iii) and for which $\|u\|_1 = 1$ and $\|u\|_\rho \leqslant \varepsilon$. We then set

$$m(\varepsilon) = m(\varepsilon,\rho,r,n) = \sup_{u\in H_\varepsilon} \|u\|_r = \sup_{u\in H_\varepsilon} u(re_1). \tag{3.1}$$

Theorem 1.2 will be proved if we show that there is a constant $\alpha \in (0,1)$ (depending on ρ, r and n) such that

$$m(\varepsilon) \leqslant \varepsilon^\alpha, \quad 0 < \varepsilon \leqslant 1. \tag{3.2}$$

Lemma 2.2 gives a useful first majorant for $m(\varepsilon)$.

PROPOSITION 3.2. *For $0 < \varepsilon \leqslant 1$,*

$$m(\varepsilon) \leqslant m_1(\varepsilon) \stackrel{\text{def}}{=} \begin{cases} \dfrac{r^{2-n}-1}{\rho^{2-n}-1}\varepsilon + \dfrac{\rho^{2-n}-r^{2-n}}{\rho^{2-n}-1} & \text{if } n \geqslant 3, \\[2ex] \dfrac{\log r}{\log\rho}\varepsilon + \dfrac{\log\rho - \log r}{\log\rho} & \text{if } n = 2. \end{cases}$$

The precise form of $m(\varepsilon)$ for 'large' ε will be determined in Section 5. However, for the proof of Theorem 1.2, we need a better majorant for small ε, one that tends to zero sufficiently rapidly as $\varepsilon \downarrow 0$.

PROPOSITION 3.3. *For $0 < \varepsilon \leqslant 1$,*

$$m(\varepsilon) \leqslant m_2(\varepsilon) \stackrel{\text{def}}{=} \sqrt{2}(1-r)^{-(n-1)/2}\varepsilon^{\beta_H/2}, \quad \beta_H = \beta_H(\rho,r) = \frac{\log r}{\log\rho}.$$

Proof. Let u be in $H(\varepsilon,\rho,r,n)$. In order to estimate $u(re_1)$, we use the Poisson integral (2.1) with $R = t\in(r,1]$. Setting $x = re_1$ and $y = t\eta$ with $\eta\in S$, $\eta = (\cos\theta, \sin\theta\cdot\eta')$, where η' runs over the unit sphere S' in $\mathbf{R}^{n-1}$, the Cauchy–Schwarz inequality shows that

$$\begin{aligned} u(re_1) &= (t^2-r^2)\,t^{n-2}\frac{1}{\sigma_n}\int_{S(0,1)}\frac{u(t\eta)}{(t^2+r^2-2tr\cos\theta)^{n/2}}d\sigma(\eta) \\ &\leqslant (t^2-r^2)\,t^{n-2}\left(\frac{1}{\sigma_n}\int_S\frac{d\sigma(\eta)}{(t^2+r^2-2tr\cos\theta)^n}\cdot\frac{1}{\sigma_n}\int_S|u(t\eta)|^2\,d\sigma(\eta)\right)^{1/2} \\ &= (t^2-r^2)\,t^{n-2}\left(\frac{\sigma_{n-1}}{\sigma_n}\int_0^\pi\frac{\sin^{n-2}\theta}{(t^2+r^2-2tr\cos\theta)^n}d\theta\right)^{1/2}\|u\|_{t,2}. \end{aligned} \tag{3.3}$$

The final integral may be evaluated with the aid of Lemma 2.4; taking $a = t^2+r^2$ and $b = 2tr$, one has $a^2-b^2 = (t^2-r^2)^2$. The L^2-norm of u on $S(0,t)$ may be estimated by Lemma 2.1, where we now take $R = 1$. Observe that $\|u\|_{s,2}$ is majorized by the supremum norm $\|u\|_s$, so that $\|u\|_{\rho,2}\leqslant\varepsilon$ and $\|u\|_{1,2}\leqslant 1$, hence $\|u\|_{t,2}\leqslant\varepsilon^\beta$ with $\beta = \beta_H(\rho,t)$. Thus (3.3) leads to the inequality

$$\begin{aligned} u(re_1) &\leqslant (t^2-r^2)\,t^{n-2}\left(\frac{t^2+r^2}{(t^2-r^2)^{n+1}}\right)^{1/2}\varepsilon^\beta \\ &\leqslant \sqrt{2}\left(\frac{t^2}{t^2-r^2}\right)^{(n-1)/2}\varepsilon^\beta,\quad \beta = \frac{\log t}{\log\rho}. \end{aligned} \tag{3.4}$$

The choice $t = \sqrt{r}$ proves the proposition.

Remark. More precise analysis of (3.4) would lead to inequalities such as

$$m(\varepsilon)\leqslant\sqrt{2}\varepsilon^{\beta_H}\left(\frac{2e}{n-1}\frac{\log\varepsilon}{\log\rho}\right)^{(n-1)/2},\quad 0<\varepsilon\leqslant\rho^\mu,\quad \mu = \frac{n-1}{1-r^2}. \tag{3.5}$$

Proof of Theorem 1.2. Fix $0<\rho<r<1$ and $n\geqslant 2$. As observed above, it is enough to prove inequality (3.2) for some constant $\alpha\in(0,1)$. To that end, let $M(\varepsilon)$ denote the infimum of the majorants for $m(\varepsilon)$ obtained in Propositions 3.2 and 3.3:

$$M(\varepsilon) = \begin{cases} m_2(\varepsilon) = c_2\varepsilon^{\beta_H/2} & \text{for } 0<\varepsilon\leqslant\varepsilon_0, \\ m_1(\varepsilon) = c_1\varepsilon+1-c_1 & \text{for } \varepsilon_0\leqslant\varepsilon\leqslant 1. \end{cases} \tag{3.6}$$

Here $c_2>1$ and $0<c_1<1$, while ε_0 is such that $m_1(\varepsilon_0) = m_2(\varepsilon_0)$. One may now determine $\alpha\in(0,\frac{1}{2}\beta_H)$ such that $\varepsilon_0^\alpha = M(\varepsilon_0)$. It is easy to verify that the resulting function ε^α is a majorant for $M(\varepsilon)$ on $[0,1]$.

4. *Proof of Theorem* 1.1

Let Ω, Ω_0 and E be subsets of $\mathbf{R}^n$ as in Theorem 1.1, and let u be harmonic on Ω. Aiming for a constant $\alpha\in(0,1)$, we may assume that u is bounded and that $\|u\|_\Omega\leqslant 1$. One may next assume that Ω is not all of $\mathbf{R}^n$ (otherwise u will be constant and there is nothing to prove).

We now enclose the compact set E in a finite union $\Omega_p = \Omega_0\cup B_1\cup\ldots\cup B_p$, where $B_1\subset\subset\Omega$ (that is, $\bar{B}_1\subset\Omega$) is a ball with centre in Ω_0, and, in general, $B_k\subset\subset\Omega$ is a ball with centre in $\Omega_{k-1} = \Omega_0 \cup B_1 \cup \ldots \cup B_{k-1}$. To explain the idea of the proof, we focus on the case of B_1 and $\Omega_1 = \Omega_0 \cup B_1$. Let V_1 and W_1 be balls concentric with B_1 such that V_1 is maximal in Ω_0 and W_1 is maximal in Ω. Then by Theorem 1.2 there is a constant $\alpha_1 \in (0,1)$, depending only on the radii of B_1, V_1, W_1 and on n, such that

$$\|u\|_{B_1} \leqslant \|u\|_{V_1}^{\alpha_1} \|u\|_{W_1}^{1-\alpha_1} \leqslant \|u\|_{\Omega_0}^{\alpha_1}.$$

Since $\|u\|_{\Omega_0} \leqslant 1$, also

$$\|u\|_{\Omega_0} \leqslant \|u\|_{\Omega_0}^{\alpha_1}, \quad \text{hence} \quad \|u\|_{\Omega_1} \leqslant \|u\|_{\Omega_0}^{\alpha_1}.$$

In the next step, one similarly proves

$$\|u\|_{\Omega_2} \leqslant \|u\|_{\Omega_1}^{\alpha_2} \leqslant \|u\|_{\Omega_0}^{\alpha_1 \alpha_2}.$$

Thus, finally, with constants $\alpha_k \in (0,1)$ depending only on the geometry,

$$\|u\|_E \leqslant \|u\|_{\Omega_p} \leqslant \|u\|_{\Omega_0}^{\alpha_1 \cdots \alpha_p}.$$

REMARKS. Coverings by balls are a standard device in the proof of inequalities as in Theorem 1.1. For the case of analytic functions on domains Ω in $\mathbf{C}^n$, such a proof may be found in [7, Chapter 8]. Similarly, for positive harmonic functions on $\Omega \subset \mathbf{R}^n$, the Harnack inequality for balls leads to the result

$$\|u\|_E \leqslant C\|u\|_{\Omega_0},$$

where C depends only on E, Ω_0 and Ω; compare [2].

Of course, the present method does not give much insight into the nature of the best (largest) constant α in Theorem 1.1.

5. *Extremal functions for the case of concentric balls*

It would be interesting to determine the largest possible constant α in Theorem 1.2. To that end, one would like to have a precise description of the function $m(\varepsilon)$ introduced in Section 3. A partial solution is given in Theorem 5.4. The corresponding extremal problem for the case of positive harmonic functions has been solved completely; see Theorem 5.1.

Let $H_\varepsilon^+ = H^+(\varepsilon, \rho, r, n)$ be the subclass consisting of the *nonnegative* harmonic functions u in H_ε. In particular, again $\|u\|_\rho \leqslant \varepsilon$ and $\|u\|_1 = 1$. We define

$$m^+(\varepsilon) = m^+(\varepsilon, \rho, r, n) = \sup_{u \in H_\varepsilon^+} \|u\|_r = \sup_{u \in H_\varepsilon^+} u(re_1), \tag{5.1}$$

$$\alpha_M^+ = \alpha_M^+(\rho, r, n) = \max\{\alpha : m^+(\varepsilon) \leqslant \varepsilon^\alpha,\ 0 < \varepsilon \leqslant 1\}. \tag{5.2}$$

The following theorem contains precise formulas for $m^+(\varepsilon)$ and α_M^+. For the formulation of the results, we introduce the spherical caps

$$S_\phi = \{y \in S \subset \mathbf{R}^n : y_1 > \cos\phi\}. \tag{5.3}$$

THEOREM 5.1. *There is a unique extremal function in the class H_ε^+. It is given by the harmonic measure $\omega_\varepsilon(x) = \omega(x, \varepsilon, \rho, n)$ (relative to the unit ball) of that spherical cap S_ϕ, $\phi = \phi_\varepsilon$, for which $\omega_\varepsilon(\rho e_1) = \varepsilon$. Thus $u(re_1) \leqslant \omega_\varepsilon(re_1)$ for all functions u in H_ε^+. In terms of the function Q given by formula* (2.5),

$$m^+(\varepsilon) = \omega_\varepsilon(re_1) = \int_0^{\phi_\varepsilon} Q(r,\theta)\, d\theta, \quad \int_0^{\phi_\varepsilon} Q(\rho,\theta)\, d\theta = \varepsilon. \tag{5.4}$$

Furthermore,

$$\alpha_M^+ \quad \textit{is equal to} \quad \gamma \stackrel{\text{def}}{=} \inf_{0<\phi<\pi} R(\phi), \tag{5.5}$$

where

$$R(\phi) = \frac{\log \int_0^\phi Q(r,\theta)\,d\theta}{\log \int_0^\phi Q(\rho,\theta)\,d\theta}. \tag{5.6}$$

Proof. In order to verify that the harmonic measure ω_ε belongs to the class H_ε^+, we shall show that on every closed ball $\bar{B}(0,t)$ with $0 < t < 1$, ω_ε assumes its maximum at the point te_1 and its minimum at the point $-te_1$. Indeed, at the points x of the sphere $S(0,t)$, the value $\omega_\varepsilon(x)$ is proportional to the average of $1/|x-y|^n$ as y runs over the cap S_ϕ, $\phi = \phi_\varepsilon$. This average depends only on the angle θ between the (positive) symmetry axis of S_ϕ and the vector x; $\omega_\varepsilon(x)$ will decrease monotonically as θ runs from 0 to π. This may be seen geometrically by keeping x fixed at the point te_1, while turning the symmetry axis of the spherical cap in the (x_1, x_2)-plane from angle θ with the positive x_1-axis to angle $\theta+d\theta$ $(0 \leqslant \theta < \theta+d\theta \leqslant \pi)$. Neighbouring caps $\Sigma(\theta)$ and $\Sigma(\theta+d\theta)$ will have a large overlap; the points y in this overlap contribute the same amount to the two harmonic measures at $x = te_1$. For the remaining parts of the caps, the points y of $\Sigma(\theta+d\theta)-\Sigma(\theta)$ will be further from $x = te_1$ than are the corresponding points y of $\Sigma(\theta)-\Sigma(\theta+d\theta)$, hence the former make a smaller contribution to the harmonic measure at te_1 than do the latter.

We shall finally prove the *basic inequality*

$$u(re_1) \leqslant \omega_\varepsilon(re_1), \quad \rho < r < 1, \quad \text{for all } u \in H_\varepsilon^+. \tag{5.7}$$

To this end we introduce the difference $w = u - \omega_\varepsilon$. Since $u \in H_\varepsilon^+$ has (nontangential) boundary values in $[0,1]$ a.e. on S, while ω_ε has boundary values 1 on S_{ϕ_ε} and 0 on the complementary (open) cap,

$$\tilde{w}(\theta) = w(\cos\theta, \sin\theta\cdot e_2)\begin{cases} \leqslant 0 & \text{a.e. for } 0 \leqslant \theta < \phi_\varepsilon, \\ \geqslant 0 & \text{a.e. for } \phi_\varepsilon < \theta \leqslant \pi. \end{cases} \tag{5.8}$$

We also know that $w(\rho e_1) = u(\rho e_1) - \varepsilon \leqslant 0$.

For $r > \rho$, we now use Lemma 2.3 to write

$$w(re_1) = \int_0^\pi \frac{Q(r,\theta)}{Q(\rho,\theta)} Q(\rho,\theta)\,\tilde{w}(\theta)\,d\theta. \tag{5.9}$$

Observe that

$$g(\theta) \stackrel{\text{def}}{=} \frac{Q(r,\theta)}{Q(\rho,\theta)} = \frac{1-r^2}{1-\rho^2}\left(\frac{1+\rho^2-2\rho\cos\theta}{1+r^2-2r\cos\theta}\right)^{n/2} \tag{5.10}$$

is positive and monotone decreasing for $0 \leqslant \theta \leqslant \pi$. Hence in view of (5.8),

$$w(re_1) = \int_0^\pi (g(\theta)-g(\phi_\varepsilon))\,Q(\rho,\theta)\,\tilde{w}(\theta)\,d\theta + g(\phi_\varepsilon)\,w(\rho e_1) \leqslant 0. \tag{5.11}$$

This formula implies the basic inequality (5.7), which proves that ω_ε is an extremal function for the class H_ε^+. The extremal function is unique: if $u \in H_\varepsilon^+$ is extremal and $w = u - \omega_\varepsilon$ so that $w(re_1) = 0$, then the integral in (5.11) must be equal to 0, hence $\tilde{w}(\theta) = 0$ a.e. on $[0,\pi]$, $w(y) = 0$ a.e. on S and thus $w(x) = 0$ throughout $B(0,1)$.

At this point formula (5.4) follows immediately from the definitions of $m^+(\varepsilon)$ and $\omega_\varepsilon(x)$, combined with Lemma 2.3.

For the proof of (5.5), we observe first that by (2.5) and the preceding,

$$0 < \int_0^\phi Q(s,\theta)\,d\theta < 1, \quad 0 < \phi < \pi,$$

$$Q(s,\theta) = Q_0(s,\theta)\sin^{n-2}\theta \quad \text{with } Q_0(s,\theta) = \frac{\sigma_{n-1}}{\sigma_n}\frac{1-s^2}{(1+s^2-2s\cos\theta)^{n/2}}.$$

It follows that $R(\phi)$ is a positive continuous function for $0 < \phi < \pi$, with the following behaviour at the end points:

$$R(0+) = \lim_{\phi\downarrow 0}\frac{\log Q_0(r,0)\,\phi^{n-1}/(n-1)}{\log Q_0(\rho,0)\,\phi^{n-1}/(n-1)} = 1,$$

$$R(\pi-) = \lim_{\phi\uparrow\pi}\frac{\log(1-\int_\phi^\pi Q(r,\theta)\,d\theta)}{\log(1-\int_\phi^\pi Q(\rho,\theta)\,d\theta)} = \frac{Q_0(r,\pi)}{Q_0(\rho,\pi)} = \frac{1-r}{1-\rho}\left(\frac{1+\rho}{1+r}\right)^{n-1} < 1. \tag{5.12}$$

Thus $\gamma = \inf_{0<\phi<\pi} R(\phi) \in (0,1)$. Finally, (5.4) and (5.6) show that

$$\frac{\log m^+(\varepsilon)}{\log\varepsilon} = R(\phi_\varepsilon) \geqslant \gamma \quad \text{or} \quad m^+(\varepsilon) \leqslant \varepsilon^\gamma, \quad 0 < \varepsilon \leqslant 1.$$

In the final inequality, no exponent α larger than the minimum γ of $R(\phi)$ on $[0,\pi]$ can work for all values of ε, since ϕ_ε runs from 0 to π as ε runs from 0 to 1. Hence $\alpha_M^+ = \gamma$.

COROLLARY 5.2. *One has*

$$\alpha_M^+(\rho,r,n) > \beta_H^n, \quad \beta_H = \frac{\log r}{\log\rho}.$$

In particular, $\alpha_M^+ \to 1$ *as* $r\downarrow\rho > 0$. *Also, by* (5.12), $\alpha_M^+ \to 0$ *as* $r\uparrow 1$.

Indeed, if $\min R$ is assumed inside $[0,\pi]$, then in obvious notation, using (5.6), calculus and (5.10),

$$\min R = \min\frac{\log F}{\log G} \geqslant \min\frac{F'}{G'}\min\frac{G}{F}$$

$$\geqslant \min\frac{F'}{G'}\min\frac{G'}{F'} = \min g/\max g = \left(\frac{(1-r)(1+\rho)}{(1+r)(1-\rho)}\right)^n > \left(\frac{\log r}{\log\rho}\right)^n.$$

By (5.12), the same lower bound for R holds at the end points.

Theorem 5.1 immediately gives an inequality for $m(\varepsilon)$.

COROLLARY 5.3. *For* $0 < \varepsilon \leqslant 1$,

$$m(\varepsilon) = m(\varepsilon,\rho,r,n) \leqslant 2m^+(\tfrac{1}{2}(1+\varepsilon),\rho,r,n) - 1. \tag{5.13}$$

Indeed, for $u \in H(\varepsilon,\rho,r,n)$, the function $\tilde{u} = \frac{1}{2}(u+1)$ belongs to the class $H_\varepsilon^+(\frac{1}{2}(1+\varepsilon),\rho,r,n)$, hence

$$\tilde{u}(re_1) \leqslant m^+(\tfrac{1}{2}(1+\varepsilon)), \quad u(re_1) \leqslant 2m^+(\tfrac{1}{2}(1+\varepsilon)) - 1.$$

The following theorem shows that the equals sign holds in (5.13) when $\varepsilon_1 \leqslant \varepsilon \leqslant 1$, where

$$\varepsilon_1 = \left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi}\right) Q(\rho,\theta)\,d\theta \quad \left(\approx \frac{2n\sigma_{n-1}}{(n-1)\sigma_n}\rho \quad \text{as } \rho \to 0\right). \tag{5.14}$$

THEOREM 5.4. *For $\varepsilon_1 \leqslant \varepsilon \leqslant 1$, the extremal function for the class $H_\varepsilon = H(\varepsilon,\rho,r,n)$ is the bounded harmonic function v_ε on the unit ball, with boundary values 1 on the spherical cap S_ψ, $\psi = \psi_\varepsilon$, and boundary values -1 on the complementary (open) cap, where ψ_ε is determined by the condition $v_\varepsilon(\rho e_1) = \varepsilon$. In terms of the function Q of (2.5),*

$$m(\varepsilon) = v_\varepsilon(re_1) = \int_0^\pi Q(r,\theta)\operatorname{sgn}(\psi_\varepsilon - \theta)\,d\theta,$$

$$\int_0^\pi Q(\rho,\theta)\operatorname{sgn}(\psi_\varepsilon - \theta)\,d\theta = \varepsilon. \tag{5.15}$$

Proof. The proof is similar to the proof of Theorem 5.1, hence we mention only the differences. Let ε_1 be defined by (5.14). By (2.4), (2.5), $Q \geqslant 0$, $\int_0^\pi Q(\rho,\theta)\,d\theta = 1$, $Q(\rho,\theta) > Q(\rho,\pi-\theta)$ for $0 < \theta < \frac{1}{2}\pi$, hence in particular $0 < \varepsilon_1 < 1$. Also,

$$\left(\int_0^{\psi} - \int_{\psi}^{\pi}\right) Q(\rho,\theta)\,d\theta, \quad 0 \leqslant \psi \leqslant \pi,$$

will be an increasing function of ψ. Taking $\varepsilon_1 \leqslant \varepsilon \leqslant 1$, we may uniquely determine $\psi_\varepsilon \in [\frac{1}{2}\pi, \pi]$ by (5.15).

Now let v_ε be the harmonic function described in the proposition, so that $\frac{1}{2}(v_\varepsilon + 1)$ is the harmonic measure of the spherical cap S_{ψ_ε}. Comparing values at ρe_1, we conclude that

$$v_\varepsilon(x) = 2\omega(x, \tfrac{1}{2}(1+\varepsilon)) - 1.$$

It follows that on the closed ball $\bar{B}(0,t)$, v_ε assumes its maximum at the point te_1 and its minimum at the point $-te_1$. Since $\psi_\varepsilon \geqslant \frac{1}{2}\pi$, $|v_\varepsilon(-te_1)| \leqslant |v_\varepsilon(te_1)|$, hence $\|v_\varepsilon\|_t = v_\varepsilon(te_1)$. In particular, $\|v_\varepsilon\|_\rho = \varepsilon$ and $\|v_\varepsilon\|_r = v_\varepsilon(re_1)$, so that v_ε belongs to the class H_ε. The basic inequality

$$u(re_1) \leqslant \|v_\varepsilon\|_r, \quad \rho < r < 1, \quad \text{for all } u \in H_\varepsilon$$

now follows from (5.13).

COROLLARY 5.5. *As ε runs from ε_1 to 1, $m(\varepsilon)$ is strictly increasing and concave: $m'(\varepsilon) = g(\psi_\varepsilon)$ decreases from $m'(\varepsilon_1) = g(\frac{1}{2}\pi)$ to*

$$m'(1) = g(\pi) = \frac{1-r}{1-\rho}\left(\frac{1+\rho}{1+r}\right)^{n-1}.$$

The last number will be an upper bound for

$$\alpha_M = \max\{\alpha : m(\varepsilon) \leqslant \varepsilon^\alpha,\ 0 < \varepsilon \leqslant 1\}.$$

In particular, $\alpha_M \to 0$ as $r \uparrow 1$. Also, $\alpha_M \to 1$ as $r \downarrow \rho > 0$.

The formula for $m'(\varepsilon)$ follows from (5.15):

$$\frac{dm}{d\varepsilon} = \frac{dm}{d\psi_\varepsilon}\bigg/\frac{d\varepsilon}{d\psi_\varepsilon} = \frac{Q(r, \psi_\varepsilon)}{Q(\rho, \psi_\varepsilon)} = g(\psi_\varepsilon);$$

compare (5.10). Finally, the formulas $m(1) = 1$ and $m(\varepsilon) \leqslant \varepsilon^\alpha$ for ε near 1 imply $\alpha \leqslant m'(1)$. A lower bound for α_M may be obtained from the majorant $M(\varepsilon)$ of $m(\varepsilon)$ in (3.6), or from an improved majorant from (3.5) and (5.13). These inequalities show that $\alpha_M \to 1$ as $r \downarrow \rho > 0$.

However, we have no lower bound for α_M similar to the lower bound for α_M^+ in Corollary 5.2.

NOTE ADDED IN PROOF. The authors have learned recently that there is an older three-balls theorem which has an additional constant C on the right-hand side of (1.3). See E. M. Landis, 'A three-spheres theorem' (Russian), *Dokl. Akad. Nauk SSSR* 148 (1963) 277–279; English translation in *Soviet Math. Dokl.* 4 (1963) 76–78.

References

1. D. H. ARMITAGE, T. BAGBY and P. M. GAUTHIER, 'Note on the decay of solutions of elliptic equations', *Bull. London Math. Soc.* 17 (1985) 554–556.
2. D. GILBARG and N. S. TRUDINGER, *Elliptic partial differential equations of second order*, Grundlehren Math. Wiss. 224 (Springer, Berlin, 1977).
3. I. S. GRADSHTEYN and I. M. RYZHIK, *Table of integrals, series and products*, corrected and enlarged edition (translated from the Russian) (Academic Press, New York, 1980).
4. W. K. HAYMAN and P. B. KENNEDY, *Subharmonic functions*, Vol. 1, London Math. Soc. Monographs 9 (Academic Press, London, 1976).
5. S. JANSON and J. PEETRE, 'Harmonic interpolation', *Interpolation spaces and allied topics in analysis*, Lecture Notes in Math. 1070 (ed. Lund, Springer, Berlin, 1984) 92–124.
6. J. L. H. MEYERS, 'Chebyshev-type quadrature formulas and potential theory', PhD thesis, University of Amsterdam, 1992.
7. R. NARASIMHAN, *Several complex variables*, Chicago Lectures in Math. (University of Chicago Press, 1971).
8. R. NEVANLINNA, *Analytic functions* (translated from the German), Grundlehren Math. Wiss. 162 (Springer, New York, 1970).
9. J. PEETRE and P. SJÖLIN, 'Three-line theorems and Clifford analysis', *Complex Variables* 19 (1992) 151–163.
10. E. D. SOLOMENTSEV, 'A three-spheres theorem for harmonic functions' (Russian), *Akad. Nauk Armyan. SSR Dokl.* 42 (1966) 274–278.

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