Introduction

Monday, September 16, 2019 1:57 PM

In the 80's, S.T. Yau conjectured that, given an d-dimensional compact smooth Riemannian manifold, M, and the Laplace operator, Δ_M, the nodal sets, $E_\lambda := \{u_\lambda = 0\}$, of the eigenfunctions u_λ satisfying

$$(*) \quad \Delta u_\lambda + \lambda u_\lambda = 0$$

satisfy the following estimate

$$c_1\sqrt{\lambda} \le \mathcal{H}^{d-1}(E_\lambda) \le C_2\sqrt{\lambda}$$

(Note that the spectrum of Δ_M is $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \quad \lambda_n \to \infty$.).

Aside:

In fact consider the heat equation on M: $(\partial_t - \Delta_M)f = 0$

The heat kernel, P_t satisfies

$$(\partial_t - \Delta_M)(P_t u_0) = 0 \quad (1)$$

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and $\quad P_t u_0 \xrightarrow{t\to 0} u_0$

P_1 is a self-adjoint operator $\quad \mu_{j,t} \xrightarrow{j\to\infty} 0$.

$\Rightarrow$ Hilbert-Schmidt $\Rightarrow$ $\exists\, \mu_j$ s.t. $P_t \varphi_j = \mu_{j,t}\varphi_j$

the semi-group property of P_t i.e. $P_t \circ P_s = P_{t+s} = P_s \circ P_t$

implies $\mu_{j,t} = \mu_{j,0}^{t}$, then (1) implies $\lambda_j = \log \mu_{j,0}$

1.) Yau's conjecture makes sense:

Example:

Consider the following compact manifold:

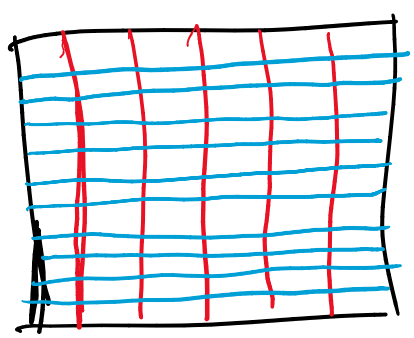
$$\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$$

and the eigenfunctions

$$u_m = \sin(m_1\theta_1)\sin(m_2\theta_2) \qquad m=(m_1,m_2)$$

Then $\lambda = \|m\|^2$.

and



then

then

$$\mathcal{H}^1(E_m) \sim m_1 + m_2 \sim \|m\|_1 = \sqrt{\lambda}$$

So Yau seems to be correct.

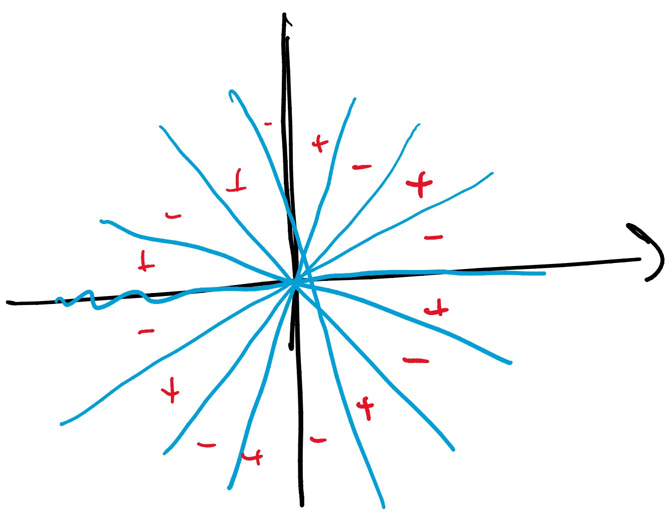
<u>Example</u>: There are some conflicting behavior for Laplace eigenfunctions on non-compact manifolds.

Consider

$$f_n(r,\theta) = r^n \sin(n\theta) \quad \text{on } \mathbb{R}^2.$$

Then $\Delta f_n = \left((\partial_r)^2 + \frac{1}{r}(\partial_r) + \frac{1}{r^2}(\partial_\theta)^2\right) f_n = 0.$

but



$\mathbb{R}^2$

$$\mathcal{H}^1(B_r \cap E_{f_n}) \sim nr.$$

This will be an important / motivating example.

I [illegible] this example would lead one

In fact, this example would lead one to believe that the control over the size of the nodal sets are solely due to global arguments. However, we will see that a common argument method finds similarities between the local behavior of

f_n to $u_{\sqrt{n}}$ as opposed to

f_n to u_0 (although these are both harmonic)

In particular, one can see that

$$\frac{\left(\int_{B_{2r}(0)} |f_n|^2\right)^{1/2}}{\left(\int_{B_r(0)} |f_n|^2\right)^{1/2}} \sim 2^n \quad , \quad \frac{\sup_{B_{2r}(0)} |f_n|}{\sup_{B_r(0)} |f_n|} \sim 2^n$$

and (from Donnelly-Fefferman).

$$\frac{\int_{B_{2r}} |u_\lambda|^2}{\int_{B_r} |u_\lambda|^2} \sim 2^{\sqrt{\lambda}} \quad , \quad \frac{\sup_{B_{2r}} |u_\lambda|}{\sup_{B_r} |u_\lambda|} \sim 2^{\sqrt{\lambda}}$$

Main Results / History

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We will present the results of a series of three papers:

1.) "Nodal sets of Laplace Eigenfunctions: Estimates of the Hausdorff Measure in Dimensions Two and Three"

A. Logunov, E. Malinnikova.

2.) "Nodal sets of Laplace Eigenfunctions: Polynomial Upper Estimates of the Hausdorff measure"

A. Logunov

3.) "Nodal sets of Laplace Eigenfunctions: Proof of Nadirashvili's conjecture and of to lower bound in Yau's conjecture".

A. Logunov.

In the first paper, the results are weaker and of smaller scope but it gives a good walkthrough of the arguments of the following papers. The summary of the results is as follows results is as follows

<u>Thm:</u> (Logunov - Malinnikova)
$\exists \alpha = \alpha(d) > 1/2$ s.t.

$$C_1 \lambda^{1/2} \leq \mathcal{H}^{d-1}(E_\lambda) \leq C_2 \lambda^\alpha.$$

where $C_1 = C_1(M)$ $C_2 = C_2(M)$.

History

1.) <u>Brüning (1978)</u> "Über Knoten von Eigenfunktionen des Laplace-Beltrami Operators"

Lower bound for $d=2$

- Variational approach.

2.) <u>Donnelly - Fefferman</u> (1988) "Nodal sets of eigenfunctions on Riemannian Manifolds"

Assume M has real-analytic metric and real-analytic chart, then Yau's conjecture holds.

- Uses tool from unique continuation

Specifically study the order of vanishing in the mold of Aronszajn (1957).

<u>Thm</u> (Aronszajn) If $\Delta u = 0$ in $\Omega \subset \mathbb{R}^d$

then $\int_{B_r(x)} |u| = O(r^\alpha)$ for all $\alpha > 0$ $\exists x \in \Omega$

then $\displaystyle\fint_{B_r(x)} |u| = O(r^{\alpha})$ for all $\alpha > 0$
$\exists x \in \Omega,$

iff $u \equiv 0$ in Ω.

- Most importantly/usefully, Donnelly-Fefferman prove a result on order of vanishing for eigenfunctions.

Thm: (Donnelly-Fefferman)

$$\frac{\sup_{B_{2r}(x)} |u_\lambda|}{\sup_{B_r(x)} |u_\lambda|} \leq 2^{c\sqrt{\lambda}}.$$

- How do they use real-analyticity?

When M is real-analytic,

$u_\lambda \sim P_{\sqrt{\lambda}} \leftarrow$ poly of degree $C\sqrt{\lambda}$

Consider $P_{\sqrt{\lambda}}$ on $B_1(0) \subset \mathbb{R}^d$

consider $G(1,d)$ the grassmanian of lines

for almost every $L \in G(1,d)$

$P_{\sqrt{\lambda}}\big|_L$ has at most $C\sqrt{\lambda}$ zeros

so

so

$$\mathcal{H}^{d-1}(\{P_{\sqrt{\lambda}}=0\}) \leq \int_{G(1,d)} \#(L \cap \{P_{\sqrt{\lambda}}=0\})\, \mu(dL)$$

$$\lesssim \sqrt{\lambda}.$$

Methods have improved since this paper and upper bounds are proved using different techniques in Hardt-Simon and Logunov-Malinnikova

For the lower bound, we have an argument that is much more popular. The argument in Donnelly-Fefferman is similar to Brüning's argument.

First, we note that zeroes of u_λ are $\sim \lambda^{-1/2}$ dense in M.

We will discuss why this is true later.
Because of this, it suffices to show that $\mathcal{H}^{d-1}(B_{\lambda^{-1/2}}(x_i) \cap E_\lambda) \gtrsim \lambda^{\frac{-d-1}{2}}$

Consider $g_\lambda(t,x) := e^{-\lambda^{1/2} t} u_\lambda(x)$ on $M \times \mathbb{R}$

then g is harmonic and

then g_λ is harmonic and

$$\{g_\lambda = 0\} = \mathbb{R} \times E_\lambda$$

By Mean Value property,

$$\int_{B_+} g_\lambda = \int_{B_-} g_\lambda$$

$\Rightarrow$ either i.) $|B_+| \sim |B_-|$

ii.) g_λ is "very large" on B_+

iii.) g_λ is "very large" on B_-

However, using a $\boxed{\text{3 spheres argument}}$, we can rule out ii.) and iii.) $\Rightarrow$ $|B_+| \sim |B_-|$

the isoperimetric inequality implies

$$\mathcal{H}^{d-1}(B_{\lambda^{-1/2}}(x_i) \cap E_\lambda) \gtrsim [\min(|B_+|, |B_-|)]^{\frac{d-1}{d}} \sim \lambda^{-\frac{d-1}{2}}.$$

3.) Hardt - Simon (1989)

"Nodal Sets for solutions to elliptic equations"

Assuming M is smooth they show that

$$\mathcal{H}^{d-1}(E_\lambda) \lesssim \lambda^{c\sqrt{\lambda}} \qquad c = c(M).$$

It is important here to emphasize that

It is important here to emphasize that since one can reduce Yau's conjecture to local considerations, the question can be rephrased as one in d-dimensional Euclidean space i.e.

$$\Delta_M u + \lambda u = 0 \iff a_{ij} D_i D_j u + b_i D_i u + c u = 0$$

$$\Delta_M u = 0 \iff a_{ij} D_i D_j u + b_i D_i u = 0 .$$

4.) FH Lin (1989)

"Nodal Sets of Solutions of Elliptic and Parabolic Equations".

Let M be analytic then and let u solve the heat equation

$$\partial_t u = \Delta_M u \quad \text{in} \quad M \times (0, \infty)$$

Then $\mathcal{H}^{d-1}(\{u=0\}) \leq N(t)$

where $N(t) \sim \log \dfrac{\int_{B(2)} |u|^2}{\int_{B(1)} |u|^2}$

* Hardt-Simon and Lin are the first

Hardt-Simon and Lin are the first papers to get an upper bound on the local size of nodal set w.r.t. the frequency.