Nodal Sets of Solutions of Elliptic and Parabolic Equations

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Some geometrical properties such as convexity and star-shapedness of level sets of positive solutions to elliptic or parabolic equations have been studied via various authors (see an excellent lecture set of notes by B. Kawohl, [8] and the references therein). One reason to choose star-shapedness and convexity, other than other properties, as geometrical properties, is that they can be easily described and are accessible to variational and maximum principles in analysis. Here we are interested in some general questions regarding level sets of solutions to elliptic and parabolic equations, such as the size (i.e., Hausdorff measure of appropriate dimension) and the topology of these level sets and estimates on critical point sets, etc. The study of such problems was motivated by the study of moving defects in evolution problems of harmonic maps and liquid crystals (see [10]). In [10], I have studied a model for the evolution of nematic liquid crystals. The singular set of optical axes (i.e., defects) of liquid crystal in motion can be described precisely by the nodal set of solutions to certain parabolic equations.

Recently, there were several rather interesting articles studying the nodal sets of eigenfunctions of Laplacians on a compact Riemannian manifold by Donnelly and Feffermann [2], [3] or, generally, solutions of second-order elliptic equations in Hardt and Simon [6].

The present work can also be viewed as a natural extension of [2], [3], and [6]. Our main result can be stated as:

THEOREM 4.2. *Suppose (M^n, g) is an analytic compact Riemannian manifold connected without boundary. Let u be a nonzero solution of*

$$\frac{\partial}{\partial t} u = \Delta_M u \quad \text{in} \quad M \times (0, \infty).$$

Then

$$H^{n-1}\{x \in M : u(x,t) = 0\} \leqq C(n, g, M) N(t). \tag{0.1}$$

Here $N(t)$ is defined by (4.7).

The estimate (0.1) is optimal. In fact, if $u = e^{-\lambda t}\phi(x)$ where $\phi(x)$, for $x \in M$ satisfies $\Delta_M \phi + \lambda\phi = 0$ on M, then (0.1) implies that

$$H^{n-1}\{x \in M : \phi(x) = 0\} \leqq C(n, g, M)\sqrt{\lambda}. \tag{0.2}$$

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This is the optimal upper bound for the nodal set of an eigenfunction ϕ of Δ_M on M, which was first shown by Donnelly and Feffermann in [2].

The proof of Theorem 4.2 is different from that of [2]. It is done first by an interesting reduction of heat equation to an elliptic equation (see also [9]) and then by a quantitative version of Cauchy's uniqueness theorem Lemma 4.3, and finally by reducing it to show the following:

THEOREM 3.1′. *Let g be an analytic metric on B_1, and let u be a nonzero solution of $\Delta_g u = 0$ in B_1. Then*

$$H^{n-1}\{x \in B_{1/2} : u(x) = 0\} \leqq C(n, g)N, \tag{0.3}$$

where $N = \int_{B_1} |\nabla_g u|^2 / \int_{\partial B_1} |u|^2$.

The quantity N in (0.3) (and also $N(t)$ in (0.1)) is called the frequency of u on B_1. Previously, N. Garofalo and the author have used this quantity to give a quantitative version of unique continuation theorems (see [5]). This is a simple replacement for the quantitative Carleman-type inequality as shown by Donnelly and Fefferman in [2].

Inequality (0.3) is proven by an integral geometry estimate and the following fact about analytic functions in the unit disc of the complex plane.

LEMMA 3.2. *Let $f(z)$ be a nonzero analytic function in $B_1 = \{z \in \mathbb{C} : |z| \leqq 1\}$. Then*

$$\operatorname{Card}\{z \in B_{1/2} : f(z) = 0\} \leqq c_0 N, \tag{0.4}$$

where $N = \int_{B_1} |\nabla f|^2 / \int_{\partial B_1} |f|^2$.

The paper is written as follows. In Section 1 we discuss the relationship between the vanishing order of a solution of a second-order elliptic equation and values of the corresponding frequency function. As a consequence we also show that the vanishing order of $\Delta_M \phi + \lambda \phi = 0$ in M cannot be larger than $C(n, g, M)\sqrt{\lambda}$, when (M^n, g) is $C^{1,1}$ (see [2]). In Section 2 we shall estimate the Hausdorff dimension of nodal sets and singular sets, i.e., the set of points where both u and $|\nabla u|$ vanish. The method we use is from geometric measure theory and is quite general. In Section 3 we prove Theorem 3.1′, to show how the frequency controls the size of nodal sets. Finally, in Section 4, we shall prove our main result, Theorem 4.2. Many results we have described above may also be generalized to nonanalytic cases; we refer to various remarks in our paper.

1. Vanishing Order and Frequency

Let u be a nonzero harmonic function defined in the unit ball $B_1(0)$ of $\mathbb{R}^n$. For $a \in B_1(0)$ and $0 < r \leqq 1 - |a|$, we define

$$(1.1) \qquad H(a, r) = \int_{\partial B_r(a)} u^2\, d\sigma, \qquad D(a, r) = \int_{B_r(a)} |\nabla u|^2\, dx$$

and

$$(1.2) \qquad N(a, r) = \frac{rD(a, r)}{H(a, r)}.$$

Here $B_r(a) = \{x \in \mathbb{R}^n : |x - a| < r\}$. $N(a, r)$ is called the frequency of u on the ball $B_r(a)$. The following fact was first observed by F. J. Almgren.

THEOREM 1.1. *$N(a, r)$ is a monotone nondecreasing function of $r \in (0, 1 - |a|)$, for any $a \in B_1(0)$.*

Since, by integration by parts, $(d/dr)H(a, r) = (n - 1)/rH(a, r) + 2D(a, r)$, that is

$$(1.3) \qquad \frac{d}{dr} \log \bar{H}(a, r) = \frac{2N(a, r)}{r}.$$

Here $\bar{H}(a, r) = H(a, r)/r^{n-1}$. We thus obtain

$$(1.4) \qquad \bar{H}(a, 2R) = \bar{H}(a, R)\exp\left(\int_R^{2R} \frac{2N(r)}{r}\, dr\right) \leqq 4^{N(a, 1-|a|)}\bar{H}(a, R), \quad \text{for all} \quad 0 < R < \tfrac{1}{2}(1 - |a|).$$

The latter inequality is due to the monotonicity of $N(a, r)$.

A consequence of the *doubling condition* (1.4) is the following quantitative version of the unique continuation theorem.

PROPOSITION 1.2. *The vanishing order of u at any point inside the ball $B_{1/4}(0)$ never exceeds $C(n)N(0, 1)$.*

Proof: By (1.3), (1.4) one has

$$(1.5) \qquad \bar{H}(0, \lambda R) \leqq \lambda^{2N(0,1)}\bar{H}(0, R), \quad \text{for all} \quad 0 < R \leqq \tfrac{1}{2},$$

and $1 < \lambda \leqq 2$. This implies, in particular, the vanishing order of u at the origin 0 is not larger than $N(0, 1)$.

Next we integrate (1.5) with respect to R to obtain

$$(1.6) \qquad \fint_{B_{\lambda R}(0)} u^2\, dx \leqq \lambda^{2N(0,1)} \fint_{B_R(0)} u^2\, dx, \quad 0 < R \leqq \tfrac{1}{2}, \quad 1 \leqq \lambda \leqq 2.$$

where $\fint_{B_\rho} u^2\,dx = (1/\rho^n)\int_{B_\rho} u^2\,dx$. Hence, for $a \in B_{1/4}(0)$, one has

$$\fint_{B_{3/4}(a)} u^2\,dx \leqq C(n)4^{2N(0,1)} \fint_{B_{1/2}(a)} u^2\,dx. \tag{1.7}$$

Then it is easy to see that

$$\fint_{\partial B_{5/8}(a)} u^2\,d\sigma \leqq C(n)4^{2N(0,1)} \fint_{\partial B_{1/2}(a)} u^2\,d\sigma \tag{1.8}$$

and, by (1.3) and the monotonicity of $N(a, r)$, that

$$N(a, r) \leqq N(a, \tfrac{1}{2}) \leqq CN(0, 1) + C(n) \quad \text{for} \quad 0 < r < \tfrac{1}{2}. \tag{1.9}$$

Thus the vanishing order of u at $a \in B_{1/4}(0)$ is not greater than $CN(0, 1) + C(n)$.

Finally one notices that if $N(0, 1) \leqq \epsilon(n) \ll 1$, and if $\int_{\partial B_1} u^2\,d\sigma = 1$ (which may always be assumed by a suitable normalization), then $u(a) \neq 0$ for all $a \in B_{1/4}(0)$. This is because

$$\begin{aligned} \|\nabla u\|_{L^\infty(B_{1/2})} &\leqq C(n)\epsilon_{(n)}^{1/2}, \\ u(0) &\geqq 1 - \epsilon_{(n)}^{1/2} \end{aligned} \tag{1.10}$$

are valid under these hypotheses.

This completes the proof of the proposition.

Theorem 1.1 and Proposition 1.2 are in fact valid for solutions of more general second-order elliptic equations. To be more precise, let us consider

$$Lu \equiv \sum_{i,j=1}^{n} (a^{ij}(x)u_{x_j})_{x_i} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u = 0 \tag{1.11}$$

in $B_1(0) \subset R^n$ with $u \not\equiv 0$, and coefficients verify the following assumptions:

(i) $a^{ij}(x)\xi_i\xi_j \geqq \lambda|\xi|^2, \quad \forall\xi \in R^n, \quad x \in B_1(0) \quad \text{and} \quad \lambda > 0;$

(ii) $\sum_{i,j} |a^{ij}(x)| + \sum_j |b^j(x)| + |c(x)| \leqq K_1, \qquad x \in B_1(0);$

(iii) $\sum_{i,j} |a^{ij}(x) - a^{ij}(y)| \leqq K_2|x - y|, \qquad x, y \in B_1(0);$

for some positive constants K_1, K_2.

THEOREM 1.3. (see [5]). *There are positive constants* $r_0 = r_0(n, \lambda, K_1)$, $\Lambda = \Lambda(n, \lambda, K_1, K_2)$ *such that the function* $N(r)\exp(\Lambda r)$ *is monotone nondecreasing on* $(0, r_0)$.

Here, for $a \in B_{1/2}(0)$, $H(r) = \int_{\partial B_1(a)} \mu u^2 \, d\sigma$,

$$D(r) = \int_{B_r(a)} \mu |\nabla_M u|^2 = \int_{B_r(a)} a^{ij}(x) u_{x_i} u_{x_j} \, dx \quad \text{and} \quad N(r) = \frac{rD(r)}{H(r)}.$$

Moreover, $C_1(n, \lambda, K_1) \leqq \mu(x) \leqq C_2(n, \lambda, K_1)$ for $x \in B_1(0)$. (See [5] for the details.) We also note that if $C(x) \leqq 0$ and $a = 0$, then r_0 may be taken to be 1. From the proof of Proposition 1.2, one also easily deduces the following:

COROLLARY 1.4. *Let* u *and* L *be as above. Suppose that* $\bar{H}(2R) \leqq 4^N \bar{H}(R)$ *for some* $R \leqq r_0/2$ *and* $N \geqq 1$. *Then the vanishing order of* u *at any point of* B_R *is never exceeded by* $C(n, \lambda, K_1, K_2)N$.

To end the discussion in this section, we would like to remark that the vanishing order of an eigenfunction can be estimated in terms of the eigenvalue. To see this, we let (M^n, g) be a C^2-connected Riemannian manifold and let u be an eigenfunction of Δ_g in M with eigenvalue λ. That is $\Delta_g u + \lambda u = 0$ in M. (In case $\partial M \neq \phi$ and is of class C^2, we will assume that $u = 0$ on ∂M). Consider the Riemannian manifold $\tilde{M}$ which is the cone over M with metric $\tilde{g}$ such that $d\tilde{g}^2 = dr^2 + r^2 \, dg^2(x)$ for $(r, x) \in (0, \infty) \times M$. It is clear that $\tilde{g}$ is a Lipschitz metric. Let $h(r, x) = r^\alpha u(x)$ with $\alpha = [\sqrt{4\lambda + (n-1)^2} - (n-1)]/2$, then $\Delta_{\tilde{g}} h = 0$ in $\tilde{M}$. For the harmonic function h on $\tilde{M}$ we have the corresponding frequency function $N(a, r)$. It is easy to check that $N(0, 2) \leqq c(n)\sqrt{\lambda}$.

By the arguments in the proof of Proposition 1.2, and by the C^2-property of the metric g in M, one then easily gets the vanishing order of u at any point of M is less than $C\sqrt{\lambda}$, where C is a positive constant which depends only on g (see also [2]).

Finally if ∂M is C^2-submanifold, then one may apply the argument in the proof of Theorem 2.3 below to obtain the same conclusion.

2. The Hausdorff Dimensions of Nodal and Singular Sets

The Hausdorff dimension of nodal sets of solutions of semilinear second-order elliptic equations was studied earlier by Caffarelli and Friedman (see [1]). Arguments of [1] were generalized to a more general class of second-order elliptic equations by Hardt and Simon (see [6]).

Here we want to show:

THEOREM 2.1. *Let* u *be a nonconstant solution of* (1.11). *Then the nodal set* $\{x \in B_1 : u(x) = 0\}$ *is of Hausdorff dimension less than or equal to* $(n-1)$, *and the singular set* $\{x \in B_1 : u(x) = |\nabla u(x)| = 0\}$ *is of Hausdorff dimension not* *exceeding* $(n-2)$. *If, in addition,* $c(x) \equiv 0$, *then the set* $\{x \in B_1 : |\nabla u(x)| = 0\}$ *is also of Hausdorff dimension not greater than* $(n-2)$.

Remark. A nonconstant solution u of (1.11) may have $|\nabla u(x)|$ vanishes on an open subset of B_1 if $c(x) \not\equiv 0$.

The proof of Theorem 2.1 is based on Federer's dimension reduction argument [13] and the bound on frequency functions. Since very little knowledge of u being a solution of equations of the form (1.11) is used, the present proof also has an interesting application in studying the line defects of liquid crystals (see [10] for details). Since, by an argument of [9], a suitable version of the Carleman type inequality will lead to bounds for frequency functions, the conclusions of Theorem 2.1 will remain valid also for the case that

$$
\begin{aligned}
&a^{ij}(x) \in C^\infty(B_1), && i,j = 1, \cdots, n;\\
&b^j(x) \in L^p(B_1), && j = 1, \cdots, n \quad \text{and} \quad p = \tfrac{1}{2}(3n-2);\\
&c(x) \in L^{n/2}(B_1), && (\text{see [6] and [12]}).
\end{aligned} \tag{2.1}
$$

To prove Theorem 2.1, we let $\mathcal{L}$ be the set of all second-order elliptic linear partial differential operators L of the form (1.11),

$$
L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left[a^{ij}(x)\cdot\frac{\partial}{\partial x^j}\cdot\right] + \sum_{j=1}^{n} b^j(x)\frac{\partial}{\partial x^j}\cdot + c(x),
$$

with coefficients satisfying (i), (ii), and (iii).

Let $\mathcal{F}_0 \equiv \{E \subset B_1 : E$ is the nodal set of some nonzero H^1-solution of $Lu = 0$ for some $L \in \mathcal{L}\}$. It is clear that for $E \in \mathcal{F}_0$, E is a relatively closed subset of B_1. We say $E_i \rightharpoonup E$ for a sequence of $\{E_i\} \subset \mathcal{F}_0$ and $E \in \mathcal{F}_0$ if

$$
(*)\qquad \begin{aligned}
&\text{for each } \epsilon > 0 \text{ there is a } i(\epsilon) \text{ such that}\\
&B_{1-\epsilon} \cap E_i \subset \{x \in B_1 : \operatorname{dist}\{E, x\} < \epsilon\} \quad \forall i \geqq i(\epsilon).
\end{aligned}
$$

Now we want to verify the following two basic properties of $\mathcal{F}_0$.

(P_1) (Closure under appropriate scaling and translation). If $|y| \leqq 1-\lambda$, $0 < \lambda < 1$, and if $E \in \mathcal{F}_0$, then $E_{y,\lambda} \in \mathcal{F}_0$. Here $E_{y,\lambda} \equiv \lambda^{-1}(E - y)$.

This is because of the fact that if $Lu = 0$ in B, for some $L \in \mathcal{L}$, and u is not identically zero in B_1, then $u_{y,\lambda}(x) = u(y + \lambda x)$, $x \in B_1$ is not a zero H-function, for $|y| \leqq 1-\lambda$, $0 < \lambda < 1$, by [4]. Moreover, $u_{y,\lambda}$ is a solution of $L_{y,\lambda}u_{y,\lambda} = 0$ in B_1, where $L_{y,\lambda}$ (defined in the obvious way) $\in \mathcal{L}$. It is then easy to see $E_{y,\lambda} \equiv \{x \in B_1 : u_{y,\lambda}(x) = 0\}$, thus $E_{y,\lambda} \in \mathcal{F}_0$.

(P_2) (Existence of homogeneous degree zero "tangent set"). If $|y| < 1$, if $\{\lambda_k\} \downarrow 0$, and if $E \in \mathcal{F}_0$, then there is a subsequence $\{\lambda_{k'}\}$ and $F \in \mathcal{F}_0$ such that $E_{y,\lambda_{k'}} \rightharpoonup F$ and $F_{0,\lambda} = F$ for each $0 < \lambda < 1$.

It is in our proof of this property that we need the bound on the frequency function (see Theorem 1.3). To show (P_2), we let $u \in H'(B_1)$ be a nonzero solution of $Lu = 0$ in B_1 for some L, and such that $E = \{x \in B_1 : u(x) = 0\}$. For $|y| < 1$, and $\lambda = 1 - |y| > 0$, we let $u_{y,\lambda}(x) = u(y + \lambda x)$ for $x \in B_1$. Hence $L_{y,\lambda} u_{y,\lambda} = 0$ in B_1 for a suitable $L_{y,\lambda} \in \mathcal{L}$ which is obtained from L by a translation and a dilation in coefficients, and $u_{y,\lambda} \neq 0$.

Without loss of generality, we may assume $\lambda_k \leqq \lambda$, (since $\lambda_k \downarrow 0$) and $u_{y,\lambda}(0) = 0$. For otherwise, it is clear by the continuity of $u_{y,\lambda}$ at 0 that $E_{y,\lambda_k} = \varnothing$ (the empty set) for all sufficiently small λ_k. Then we simply take F to be the empty set and (P_2) is obviously valid.

Since $u_{y,\lambda}$ is not zero, we have, from Theorem 1.3, that

$$N(0, r) \leqq e^{\Lambda r_0} N(0, r_0) < \infty, \qquad \forall 0 < r \leqq r_0. \tag{2.2}$$

Here N is the corresponding frequency function for $u_{y,\lambda}$.

It should be pointed out that the right-hand side of (2.2) is certainly dependent on u and $y \in B_1$, $|y| < 1$. But the bound is uniform in $r \in (0, r_0)$ whenever u and y are fixed.

Now for each λ_k such that $\lambda_k \leqq r_0 \lambda$, we define $v_k(x) = u_{y,\lambda_x}(x) / \int_{\partial B_1} |u_{y,\lambda_k}|^2$. Let $L_k = L_{y,\lambda_k}$, then $L_k v_k = 0$ in B_1. It is easy to check that $L_k \to L_y = \sum_{i,j=1}^n (\partial/\partial x^i)[a^{ij}(y)(\partial/\partial x^j)\cdot]$ as $k \to \infty$ (in the sense that the corresponding coefficients converge uniformly in B_1). Since (2.2), we have

$$\overline{\lim}_k \int_{B_2} |\nabla v_k(x)|^2 \, dx \leqq c(n, \lambda, K_1, K_2) N(0, 0) < \infty \tag{2.3}$$

and, by a doubling condition similar to (1.4),

$$\int_{\partial B_{1/2}} |v_k|^2 \geqq c(n, \lambda, K_1, K_2) > 0 \qquad \forall k \text{ large}. \tag{2.4}$$

By taking a subsequence k's we may assume that $v_{k'} \xrightarrow{H'} V$ and $v'_k \xrightarrow{C^\alpha} v$ in B_2.

It is obvious then $L_y v = 0$ in B_2. Moreover

$$\lim_{k'} \int_{\partial B_{1/2}} |v'_k|^2 = \int_{\partial B_{1/2}} |v|^2 \geqq c(n, \lambda, K_1, K_2) > 0,$$

$$\lim_{k'} \int_{\partial B_1} |v_{k'}|^2 = 1, \quad v(0) = 0 \quad \text{and} \tag{2.5}$$

$$\int_{B_1} |\nabla v|^2(x) \leqq c(n, \lambda, K_1) N(0, 0) < \infty.$$

Let $F = \{x \in B_1 : v(x) = 0\} \in \mathcal{F}$. We want to check $E_{y,\lambda_{k'}} \rightharpoonup F$ and $F_{0,\lambda} = F$ for each $\lambda \in (0, 1)$.

Since $E_{y,\lambda_k'} = \{x \in B_1 : v_{k'}(x) = 0\}$ and since $v_{k'}(x) \xrightarrow{C^\alpha} v(x)$ in B_2 with $v \neq 0$, it is then clear that $E_{y,\lambda_k'} \rightharpoonup F = \{x \in B_1 : v(x) = 0\}$. Next one observes, from (2.2) and the monotonicity of $e^{\Lambda r}N(0, r)$, that the frequency function $N_k(0, 1)$ corresponding to v_k satisfies

$$N(0,0) = \lim_{r \downarrow 0} e^{\Lambda r} N(0, 1) \leqq e^{\lambda_k r \Lambda} N(0, \lambda_k r) \tag{2.6}$$
$$= e^{\lambda_k r \Lambda} N_k(0, r) \leqq e^{2\lambda_k \Lambda} N(0, 2\lambda_k)$$

for all $r \in (0, 2)$. Hence the frequency function $N_0(0, r)$ for v satisfies

$$N_0(0, r) \equiv N(0, 0) \qquad \forall r \in (0, 2). \tag{2.7}$$

It is then clear that $v(\lambda x) = \lambda^{N(0,0)} v(x)$, $0 < \lambda \leqq 1$ a homogeneous function of degree $N(0, 0)$. Hence $F_{0,\lambda} = F$ for $0 < \lambda \leqq 1$. This completes the proof of (P_2).

By [13, Appendix A] we have the following:

LEMMA 2.2. *Let $\mathcal{F}$ be a collection of relatively closed proper subsets of $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ which satisfies (P_1) and (P_2). Then*

$$\dim(B_1 \cap E) \leqq n - 1 \qquad \forall E \in \mathcal{F}. \tag{**}$$

(Here "dim" is Hausdorff dimension, so that (**) means $H^{n-1+\delta}(E) = 0$ for all $\delta > 0$.)

In fact either $E \cap B_1(0) = \emptyset$ for every $E \in \mathcal{F}$ or else there is an integer $d \in [0, n - 1]$ such that

$$\dim(E \cap B_1) \leqq d \qquad \forall E \in \mathcal{F}$$

and such that there is some $F \in \mathcal{F}$ which is a d-dimensional subspace of $\mathbb{R}^n$ with $F_{y,\lambda} = F$, for all $y \in F$, $0 < \lambda \leqq 1$.

If $d = 0$, then $E \cap B_\rho$ is finite for each $E \in \mathcal{F}$ and each $\rho < 1$.

Remark. If we define $\mathcal{F}_s = \{E \subset B_1 : E = \{x \in B_1 : u(x) = |\nabla u(x)| = 0\}$ where u is a nonzero solution of $Lu = 0$ in B_1 for some $L \in \mathcal{L}\}$, then it follows from the above arguments that $\mathcal{F}_s$ satisfies (P_1) and (P_2). In particular we may apply Lemma 2.2 to $\mathcal{F}_s$.

Proof of Theorem 2.1: If u is a nonzero solution of (1.11), then the nodal set $E = \{x \in B_1 : u(x) = 0\} \in \mathcal{F}_0$ which verifies the hypothesis of Lemma 2.2. Therefore $\dim E \leqq n - 1$. Next we let $S = \{x \in B_1 : u(x) = |\nabla u(x)| = 0\} \in \mathcal{F}_s$. We apply Lemma 2.2 to $\mathcal{F}_s$ to conclude that $\dim S \leqq d$. We only need to check that $d \neq n-1$. Since there is an $F \in \mathcal{F}_s$ which is a d-dimensional subspace of $\mathbb{R}^n$, i.e., there is a nonzero $u \in H'(B_1)$ and some $L \in \mathcal{L}$ such that $Lu = 0$ in B_1 and that $F = \{x \in B_1 : u(x) = |\nabla u(x)| = 0\}$, we have, by the uniqueness of Cauchy's problem for equation (1.11), $d \neq n-1$. We note also that if $n = 2$, then S consists of isolated points.

Finally we let u be a nonconstant solution of (1.11) with $c(x) \equiv 0$, and let $F = \{x \in B_1 : |\nabla u(x)| = 0\}$. Denote by $\mathcal{F}$ the collection of all such sets F. Since, by [5], $|\nabla u|$ is an A_p-weight for some $p > 1$, one can verify easily that $\mathcal{F}$ satisfies (P_1) and (P_2) as a doubling condition similar to (1.4) holds for $|\nabla u|$. Applying Lemma 2.2, we conclude that there is an integer $d \in [0, n-1]$ such that $\dim F \leqq d$ for all $F \in \mathcal{F}$ and that there is a d-dimensional subspace $L \in \mathcal{F}$. We apply again the uniqueness of Cauchy's problem for equation (1.11) to obtain $d \neq n-1$.

We would like to conclude this section with the following:

THEOREM 2.3. *Let u be a nonzero H'-solution of (1.11) in a $C^{1,1}$ domain Ω. Let $\Gamma \subset \partial\Omega$ be a $C^{1,1}$, $(n-1)$-dimensional submanifold of $\partial\Omega$ such that $u = 0$ on Γ. Then the set $\{x \in \Gamma : |\nabla u(x)| = 0\}$ is of Hausdorff dimension not exceeding $n-2$.*

Remark. With a little more work, one can show that the above theorem remains valid provided that $\partial\Omega$, Γ are $C^{1,\alpha}$ for some $\alpha > 0$. It is, however, an open problem when Γ and $\partial\Omega$ are only Lipschitz. In this case one does not even know if the set $\{x \in \Gamma : |\nabla u(x)| = 0\}$ is of Hausdorff $(n-1)$-dimensional measure zero.

Proof of Theorem 2.3: By a suitable changing of independent variables, one may reduce to the following situation: $u \neq 0$ is a solution of

$$
\begin{aligned}
&Lu = \frac{\partial}{\partial x^i}(a^{ij}(x)\, u_{x_j}) + b^j(x) u_{x_j} + c(x) u = 0 \quad \text{in} \quad B_+, \\
&u|_{x_n=0} = 0 \quad \text{and} \quad u \in H'(B_+) \quad \text{where} \quad B_+ = \{x \in B_1 : x_n \geqq 0\},
\end{aligned} \tag{2.8}
$$

and the coefficients of L verify conditions (i), (ii), and (iii) in (1.11).

Following the arguments of [5], we let

$$
\begin{aligned}
H(r) &= \int_{\partial B_+(0,r)} \mu u^2 = \int_{\{|x|=r\} \cap B_+} \mu u^2, \\
D(r) &= \int_{B_+(0,r)} \mu |\nabla_M u|^2 = \int_{B_+(0,r)} a^{ij}(x) u_{x_i} u_{x_j}, \quad \text{and} \\
N(r) &= \frac{rD(r)}{H(r)} \quad \text{for} \quad 0 < r \leqq r_0 < 1.
\end{aligned} \tag{2.9}
$$

(See also Section 1 of the paper.) Then one also verifies that

$$(2.10) \qquad e^{\Lambda r}N(r) \quad \text{is an increasing function of} \quad r \in (0, r_0)$$

for some positive $\Lambda = \Lambda(n, \lambda, K_1, K_2, \partial\Omega)$.

Now, as in the proof of Theorem 2.1, we consider the set

$$\mathcal{F}_0 = \{E \subset B_+ : E = \{x \in B_+ : u(x) = 0 = |\nabla u(x)|\}$$

$$\text{for some nonzero } H'\text{-solution of (2.8) with} \quad L \in \mathcal{L}\}.$$

One then verifies that (P_1) and (P_2) are valid for $\mathcal{F}_0$. Hence, by Lemma 2 and by the uniqueness of Cauchy's problem for equations (2.8) with $L \in \mathcal{L}$, we have

$$\dim(E) \leqq d \leqq n - 2 \quad \text{for each} \quad E \in \mathcal{F}_0.$$

The conclusion of Theorem 2.3 follows.

3. Controlling the Size of Nodal Sets by the Frequency

Let $P(x)$ be a degree N polynomial, $x \in \mathbb{R}^n$. Suppose $\dim\{x \in \mathbb{R}^n : P(x) = 0\} = k$. Then it is a classical fact that $H^k\{x \in B_1 : P(x) = 0\} \leqq C(n)N^{n-k}$. From the discussions in Sections 1 and 2 above, it is natural to make the following conjectures:

Conjecture 1. $H^{n-1}\{x \in B_{1/2} : u(x) = 0\} \leqq c(n, \lambda, K_1, K_2)N$.

Conjecture 2. $H^{n-2}\{x \in B_{1/2} : u(x) = |\nabla u(x)| = 0\} \leqq c(n, \lambda, K_1, K_2)N^2$. Here u is a nonzero solution of (1.11) such that

$$(3.1) \qquad N(a, r_0) \leqq N \quad \text{for all} \quad a \in B_{1-r_0}(0);$$

($r_0 = r_0(n, \lambda, K_1)$ is given in Theorem 1.3).

It was shown by Hardt and Simon in [6] that

$$(3.2a) \qquad H^{n-1}\{x \in B_{1/2} : u(x) = 0\} \leqq c \exp(c\sqrt{N} \log N).$$

Here u is a solution of (1.11), and satisfies (3.1), where c is a positive constant which depends only on n, λ, K_1, and K_2. On the other hand, Donnelly and Fefferman show that

$$(3.2b) \qquad H^{n-1}\{x \in M : u(x) = 0\} \leqq c(n, g)\sqrt{\lambda}$$

for eigenfunctions $\Delta_g u + \lambda u = 0$ on an analytic manifold (M^n, g). Here we will show that Conjecture 1 is true for solutions of (1.11) provided that all the coefficients are real analytic in B_1. More precisely we have the following:

THEOREM 3.1. *Let u be a nonzero solution of (1.11). Suppose all the coefficients of (1.11) are real analytic in B_1, and that (3.1) is valid. Then there is a positive constant C which depends on λ, n, K_1, K_2 and the analyticity of the coefficients such that*

$$H^{n-1}\{x \in B_{1/2} : u(x) = 0\} \leqq CN. \tag{3.3}$$

Remark (a). Let u be a solution of the uniformly elliptic equation

$$a^{ij}(x)u_{i_j} + b^i(x)u_i + c(x)u = 0 \quad \text{in} \quad B_1^n \subseteq \mathbb{R}^n \tag{3.4}$$

with all coefficients bounded measurable in B_1^n. For $(x, x_{n+1}) \in B_1^{n+1} \subset \mathbb{R}^{n+1}$, we define $V(x, x_{n+1}) = (2 - x_{n+1})u(x)$. Then

$$\tilde{a}^{ij}V_{i_j} + \tilde{b}^i V_i = 0 \quad \text{in} \quad B_1^{n+1} \quad \text{where}$$

$$(\tilde{a}^{ij}) = \begin{pmatrix} (a^{ij}) & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{b}^i(x) = \begin{cases} b^i(s) & \text{for} \quad 1 \leqq i \leqq n \\ -(2 - x_{n+1})c(x) & \text{for} \quad i = n+1. \end{cases} \tag{3.5}$$

Next we let $W(x, x_{n+1}, x_{n+2}) = (2 - x_{n+2})V(x, x_{n+1})$ for $(x, x_{n+1}, x_{n+2}) \in B_1^{n+2}$, and let M be a suitably large number; then

$$A^{ij}W_{i_j} = 0 \quad \text{in} \quad B_1^{n+2} \quad \text{where}$$

$$(A^{ij}) = \begin{pmatrix} (\tilde{a}^{ij}), & -\tilde{b}/2(2 - x_{n+2}) \\ -\left(\dfrac{\tilde{b}}{2}\right)^t (2 - x_{n+2}), & M^2 \end{pmatrix}. \tag{3.6}$$

Obviously equation (3.6) is also uniformly elliptic. One also notices that $W(x, x_{n+1}, x_{n+2}) = 0$ for $(x, x_{n+1}, x_{n+2}) \in B_1^{n+2}$ if and only if $u(x) = 0$ for $x \in B_1^n$.

Remark (b). Let u be a solution of the uniformly elliptic equation

$$a^{ij}(x)u_{i_j} = 0 \quad \text{in} \quad B_1^n \tag{3.7}$$

with $a^{ij} \in C^2(B_1^n)$. Then

$$\frac{\partial}{\partial x^i}\left(a^{ij}(x)\frac{\partial}{\partial x^i}u\right) - \left(\frac{\partial}{\partial x^i}a^{ij}(x)\right)u_{x_j} = 0 \quad \text{in} \quad B_1^n.$$

Therefore

$$\frac{\partial}{\partial x^i}\left(\tilde{a}^{ij}(x, x_{n+1})\frac{\partial}{\partial x^j}\bar{u}\right) = 0 \quad \text{in} \quad B_1^{n+1} \tag{3.8}$$

where $\bar{u}(x, x_{n+1}) = u(x)$, and

$$(\tilde{a}^{ij}(x, x_{n+1})) = \begin{pmatrix} (a^{ij}(x)), & x_{n+1}b \\ x_{n+1}b^t, & M^2 \end{pmatrix}$$

with $b^j = -(\partial/\partial x^i)(a^{ij}(x))$ and M a suitable large number. It is clear that (3.8) is uniformly elliptic. Because the doubling condition similar to (1.4) is equivalent to the bound on frequency functions, and because of the above remarks, we conclude that to prove Theorem 3.1 is equivalent to showing the following:

THEOREM 3.1′. *Let g be an analytic metric on B_1, and let u be a nonzero solution of $\Delta_g u = 0$ in B_1. Then*

$$H^{n-1}\{x \in B_{1/2} : u(x) = 0\} \leqq C(n, g)N \tag{3.9}$$

where $N = \int_{B_1} |\nabla_y u|^2 / \int_{\partial B_1} u^2$.

Theorem 3.1′ is proven by induction on n and an integral geometry estimate. We start with the following:

LEMMA 3.2. *Let $f(z)$ be a nonzero analytic function in $B_1 = \{z \in \mathbb{C} : |z| \leqq 1\}$. Then*

$$\operatorname{Card}\{z \in B_{1/8} : f(z) = 0\} \leqq c_0 N, \tag{3.10}$$

where $N = \int_{B_1} |\nabla f|^2 / \int_{\partial B_1} |f|^2$.

Proof. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$, for $|z| < 1$. We normalize f so that

$$\sum_{j=0}^{\infty} |a_j|^2 = \int_{\partial B_1} |f(z)|^2 = 1. \tag{3.11}$$

Fix $\delta > 0$ which will be chosen later, and let $z_1, \cdots, z_m$ be zeros of f in $B_\delta(0) = \{z \in \mathbb{C} : |z| \leqq \delta\}$. Denote by $p(z) = \prod_{j=1}^{m}(z - z_j)$ and $g(z) = f(z)/p(z) = \sum_{j=0}^{\infty} b_j z^j$. Then

$$\sum_{j=0}^{\infty} |b_j|^2 = \int_{\partial B_1} |g|^2 \leqq (1 - \delta)^{-2m}. \tag{3.12}$$

Since $a_0 = (-1)^m \prod_{j=1}^m z_j b_0$,

$$a_1 = (-1)^m \left(\prod_{j=1}^{m} z_j \cdot b_1 - \sum_{i=1}^{m} \prod_{j \neq i} z_j \cdot b_0 \right), \quad \text{etc.,}$$

we see that

$$
\begin{aligned}
&|a_0|^2 \leqq |b_0|^2 \delta^{2m}, \\
&|a_1|^2 \leqq (|b_0|^2 + |b_1|^2)(\delta^{2m} + m^2 \delta^{2m-2}), \cdots, \\
&|a_{[m/2]}|^2 \leqq \left(\sum_{j=0}^{[m2/]} |b_j|^2 \right) \left(\delta^{2m} + m^2 \delta^{2m-2} + \cdots + \binom{m}{\frac{m}{2}} \delta^m \right).
\end{aligned}
\tag{3.13}
$$

Therefore

$$
\begin{aligned}
\sum_{j=0}^{[m/2]} |a_j|^2 &\leqq m(1-\delta)^{-2m} \left(\delta^{2m} + m^2 \delta^{2m-2} + \cdots + \binom{m}{\frac{m}{2}} \delta^m \right) \\
&\leqq m \left(\frac{4\delta}{(1-\delta)^2} \right)^m .
\end{aligned}
\tag{3.14}
$$

On the other hand,

$$
\begin{aligned}
N &= \int_{B_1} |\nabla f|^2 \Big/ \int_{\partial B_1} |f|^2 = \int_{B_1} |\nabla f|^2 \qquad (\text{by } (3.11)) \\
&= \sum_j j |a_j|^2 .
\end{aligned}
\tag{3.15}
$$

Hence

$$\sum_{j \leqq 3N} |a|^2 \geqq \tfrac{2}{3}. \tag{3.16}$$

Now if $\delta \leqq \frac{1}{8}$, then (3.14) and (3.16) imply that $m \leqq \max\{c_1, c_2 N\}$. It is also clear if $N \leqq \epsilon_0$, then $f(z) \neq 0$ for $z \in B_{1/2}(0)$. We thus have $m \leqq c_0 N$.

Remark (c). Via the discussions in Section 1, we can easily generalize Lemma 3.2 to show that if

(3.17) $$\fint_{|z|=1/2} |f|^2 \geqq 4^{-N} \fint_{|z|=1} |f|^2,$$

then

(3.18) $$\operatorname{Card}\{z \in B_{3/4} : f(z) = 0\} \leqq c_1 N,$$

and

$$\operatorname{Card}\{z \in B_\rho : f(z) = 0\} \leqq C(\rho) N, \quad \text{for} \quad \rho \in (0, 1).$$

Remark (d). By [12] we see that a solution u of $\Delta_g u = 0$ in B_1 is analytic in B_1 if g is an analytic metric. Moreover, there is a $\rho_0 = \rho_0(n, g) > 0$ such that $u(x)$ extends to an analytic function $u(z)$ on

$$D = \{z \in \mathbb{C}^n : z = x + iy, \quad x \in B_{3/4}, \quad y \in (-\rho_0, \rho_0)\}$$

with $\|u(z)\|_{L^2(D)} \leqq C(n, g) \|u(x)\|_{L^2(B_1)}$.

These remarks will be needed in the proof of Theorem 3.1′.

Proof of Theorem 3.1′: We may assume that $\int_{\partial B_1} u^2 = 1$. From the discussions in Section 1, we obtain

(3.19) $$\fint_{B_\rho(a)} |u|^2\, dx \geqq 4^{-CN} \fint_{B_{2\rho}(a)} |u|^2\, dx$$

for $a \in B_{3/4}(0)$ and $0 < \rho < \frac{1}{8}$, and also that

(3.20a) $$\fint_{B_{1/10}(a)} |u|^2\, dx \geqq 4^{-CN}, \quad \text{for} \quad a \in B_{3/4}(0).$$

In particular, there is a point $x_a \in B_{1/10}(a)$ such that $|u(x_a)| \geqq 2^{-cN}$.

Now we choose $a_1, \cdots, a_n \in \partial B_{1/4}(0)$, $a_j = (\overbrace{0, 0 \cdots \tfrac{1}{4}}^{j} 0 \cdots 0)$, and let $x_{a_j} \in B_{1/16}(a_j)$ be such that $|u(x_{\mathbb{S}_j})| \geqq 2^{-cN}, j = 1, \cdots, n$.

For each j and $w \in \mathbb{S}^{n-1}$ we consider $f_j(w, t) = u(x_{u_j} + tw)$ for $t \in (-5/8, 5/8)$. It is obvious that $f_j(\omega, t)$ is an analytic function of $t \in (-5/8, 5/8)$. Moreover $f_j(\omega, t)$ extends to an analytic function $f_j(\omega, z)$ for $z = t + iy$, $|t| < 5/8$ and $|y| < \rho_0$. (See Remark (b) above.) Since $|f_j(\omega, 0)| \geqq 2^{-cN}$ and

$$|f_j(\omega, t + iy)| \leqq C(n, g) \quad \text{by Remark (d).}$$

Applying Lemma 3.2 to $f_j(\omega, \cdot)$ we see, via Remark (c), that

$$\text{(3.20b)} \quad \begin{aligned} &\operatorname{Card}\{t : u(x_{a_j} + t\omega) = 0 \quad \text{and} \\ &\qquad x_{a_j} + t\omega \in B_{1/16}(0)\} \equiv N_j(\omega) \leqq C(n, g)N. \end{aligned}$$

From an integral geometry estimate (see [4, Chapter 3]), we have

$$\text{(3.21)} \quad H^{n-1}\{x \in B_{1/16}(0) : u(x) = 0\} \leqq c(n) \sum_{j=1}^{n} \int_{\mathbb{S}^{n-1}} N_j(\omega)\, d\omega \leqq C(n, g)N.$$

Now (3.9) follows simply by a suitable finite covering of $B_{1/2}(0)$ by balls of radius 1/16. This completes the proof of Theorem 3.1′.

Remark (e). By using the same proof as for Theorem 3.1′, one can show that if

$$\text{(3.22)} \qquad f(x) = \sum_{\lambda} a_\lambda x^\lambda \quad \text{for} \quad x \in B_1^n(0),$$

and if $N = \sum_\lambda |a_\lambda|^2 |\lambda| / \sum |a_\lambda|^2 < \infty$, then $H^{n-1}\{x \in B_{1/2} : f(x) = 0\} \leqq C(n)N$.

It was shown by R. Thom and J. Milnor (see [11]) that if $f(x)$ is a polynomial of degree less than or equal to N for $x \in \mathbb{R}^n$, then the

$$\text{Total Betti number of} \quad \{f(x) = 0\} \leqq C(n)N^n.$$

It is, therefore, natural to conjecture that if an analytic function is given by (3.22), then the

$$\text{Total Betti number of} \quad \{x \in B_{1/2} : f(x) = 0\} \leqq C(n)N^n.$$

We also conjecture that if u is as in Theorem 3.1′, then the

$$\text{Total Betti number of} \quad \{x \in B_{1/2} : u(x) = 0\} \leqq C(n, g)N^n.$$

Remark (f). A generalization of Theorem 3.1′ to the case that u is defined on an analytic bounded domain Ω with analytic boundary $\partial\Omega$ is possible as in [3]. Also, the estimate (3.2) is a consequence of Theorem 3.1 and the discussions at the end of Section 1.

4. Nodal Sets of Heat Equations

In this section we will derive estimates on the size of nodal sets of solutions to a class of parabolic equations with time independent coefficients.

To start with we consider the following

$$(4.1)\qquad \begin{aligned} \frac{\partial}{\partial t} u &= \Delta_g u \quad \text{in} \quad M \times (0, \infty) \\ u(x, 0) &= u_0(x) \end{aligned}$$

where (M^n, g) is an n-dimensional connected smooth Riemannian manifold without boundary.

Let $\{\phi_j\}$ be eigenfunctions of Δ_g on M with corresponding eigenvalues $\{\lambda_j\}$ such that $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ and such that $\{\phi_j\}$ form an orthonormal basis of $L^2(M, \mathbb{R})$. Thus we may write $u_0(x)$ (which we will assume to be in $L^2(M, \mathbb{R})$ in its Fourier series:

$$(4.2)\qquad u_0(x) = \sum_{j=0}^{\infty} c_j \phi_j(x).$$

It is then easy to see, for each $t > 0$, that the solution u of (4.1) can be written as

$$(4.3)\qquad u(x, t) = \sum_{j=0}^{\infty} c_j e^{-\lambda_j t} \phi_j(x).$$

We define, for $y \in R^1$, a new function

$$(4.4)\qquad \bar{u}(x, t, y) = \sum_{j=0}^{\infty} c_j e^{-\lambda_j t} \phi_j(x) \cosh(\sqrt{\lambda_j}\, y).$$

(It is clear that the series in (4.4) converges uniformly for $|y| \leqq 10$ and each fixed $t > 0$.)

As in [9] we see that

$$(4.5)\qquad \Delta_g \bar{u} + \bar{u}_{yy} = 0 \quad \text{in} \quad M \times (-10, 10)$$

and

$$(4.6)\qquad \begin{aligned} \bar{u}(x, t, 0) &= u(x, t) \\ & \qquad\qquad \text{for} \quad (x, t) \in M \times (0, \infty). \\ \bar{u}_y(x, t, 0) &= 0 \end{aligned}$$

Let $N(t) = \log\left(\int_{M \times (-2,2)} \bar{u}^2(x, t, y)\, dx\, dy / \int_{M \times (-1,1)} \bar{u}^2(x, t, y)\, dx\, dy\right)$. A simple calculation shows

(4.7) $$N(t) = \log\left(\sum_{j=0}^{\infty} c_j^2 e^{-2\lambda_j t}\left(2 + \frac{\sinh 4\sqrt{\lambda_j}}{2\sqrt{\lambda_j}}\right) \Big/ \sum_{j=0}^{\infty} c_j^2 e^{-2\lambda_j t}\left(1 + \frac{\sinh 2\sqrt{\lambda_j}}{2\sqrt{\lambda_j}}\right)\right).$$

THEOREM 4.1. *If (M^n, g) is an analytic manifold with real analytic metric g, then*

(4.8) $$H^n\{(x, y) \in M \times (-1, 1) : \bar{u}(x, t, y) = 0\} \leqq c(n, g, M)N(t).$$

Remark. If (M^n, g) is only $C^{1,1}$, then the right-hand side of (4.8) should be replaced by $CN(t)^{C\sqrt{N(t)}}$, where $c = c(n, g, M)$. This follows from the arguments below and estimates by Hardt and Simon in [6].

Proof of Theorem 4.1: By a suitable scaling in metric g, we may assume that the injectivity radius of (M, g) is not less than two and the diam$(M) = D$. We also normalize $\bar{u}$ so that $\int_{M\times(-1,1)} \bar{u}^2(x, t, y)\, dx\, dy = 1$. Hence, for some $x_0 \in M$, one has

(4.9) $$\int_{B_1(x_0)\times(-1,1)} \bar{u}^2\, dx\, dy \geqq c(g, M) > 0,$$

and

(4.10) $$\int_{B_2(x_0)\times(-2,2)} \bar{u}^2\, dx\, dy \leqq 4^{N(t)}.$$

By the arguments in Section 1, we conclude that

(4.11) $$\int_{B_\rho(x)\times(-\rho,\rho)} \bar{u}^2\, dx\, dy \geqq \rho^{-CN(t)}$$

for all $x \in B_{5/3}(x_0)$, $\rho \in (0, \frac{1}{4})$ and for some positive constant c which depends only on g in $B_2(x_0)$.

Recall that M is connected. If $x \in M$ is arbitrary, we may join x_0 to x by an overlapping chain of balls, with radius $1/4$, whose centers are separated by a distance at most $1/8$. Using (4.9), (4.10), (4.11) inductively, we see that, for any $x^* \in M$, $\rho \in (0, 1)$,

(4.12) $$\int_{B_{1/4}(x^*)\times(-1/4,1/4)} \bar{u}^2\, dx\, dy \geqq 4^{-CN(t)},$$

and

(4.13) $$\int_{B_\rho(x^*)\times(-\rho,\rho)} \bar{u}^2\, dx\, dy \geqq \rho^{-CN(t)}.$$

Here C is a constant depending on D, the ellipticity bound of Δ_g and the Lipschitz continuity of g on any $B_1(x)$, $x \in M$.

Finally Theorem 4.1 follows from (4.12), (4.10), and Theorem 3.1′.

The main result of this section is the following:

THEOREM 4.2. *Suppose (M^n, g) is an analytic compact Riemannian manifold connected without boundary. Let u be a nonzero solution of (4.1). Then*

$$H^{n-1}\{x \in M : u(x, t) = 0\} \leqq C(n, g, M)N(t). \tag{4.14}$$

To prove Theorem 4.2, we need a quantitative Cauchy uniqueness theorem. This can be stated as:

LEMMA 4.3. *Let u be a solution of (1.11) in B_1^+ with $\|u\|_{L^2(B_+)} \leqq 1$. Suppose that $\|u\|_{H'(\Gamma)} + \|(\partial u/\partial x_n)\|_{L^2(\Gamma)} \leqq \epsilon \ll 1$, where $\Gamma = \{(x', 0) \in \mathbb{R}^n : |x'| < 3/4\}$. Then $\|u\|_{L^2(B^+_{1/2})} \leqq c\epsilon^\alpha$ for some positive constants C, α which depend only on n, λ, K_1, K_2.*

Proof of Theorem 4.2: From the proof of Theorem 4.1, we see that, for $\rho \in (0, \frac{1}{4})$, $x^* \in M$, and for some $C = C(n, g, M) > 0$,

$$\int_{B_\rho(x^*)\times(-\rho,\rho)} \bar{u}^2\, dx\, dy \geqq 4^{-CN(t)} \int_{B_{2\rho}(x^*)\times(-2\rho,2\rho)} \bar{u}^2\, dx\, dy. \tag{4.15}$$

We want to deduce the same doubling condition for $\bar{u}$ restricted to the hyperplane $y = 0$ that is, by (4.6),

$$\int_{B_\rho(x^*)} u^2(x, t)\, dx \geqq 4^{-CN(t)} \int_{B_{2\rho}(x^*)} u^2(x, t)\, dx \tag{4.16}$$

for $\rho \in (0, \frac{1}{4})$, $x^* \in M$ and a constant C which (possibly different from that in (4.15)) depends only on M and g.

Since $u(\cdot, t)$ is real-analytic, the discussion in Section 3 can easily be applied to $u(\cdot, t)$. We conclude, from (4.16), that (4.14) is true.

To show (4.16) we need Lemma 4.3. By a rescaling, we may assume $\rho = 1/2$ and $x^* = 0$ for simplicity. We normalize $\bar{u}$ so that

$$\int_{B_1(0)\times(-1,1)} \bar{u}^2\, dx\, dy = 1.$$

Hence by (4.15) we have

$$\int_{B_{1/4}(0)\times(-1/4,1/4)} \bar{u}^2\, dx\, dy \geqq 4^{-CN(t)}. \tag{4.17}$$

Let $\epsilon = \|u(\cdot, t)\|_{H'(\Gamma)} = \|\bar{u}\|_{H'(\Gamma)}$, since $(\partial\bar{u}/\partial y) = 0$ on $\Gamma = \{(x, 0) : x \in B_{3/8}(0)\}$, one has, by Lemma 4.3 and (4.17), that

$$\epsilon \geqq 4^{-CN(t)}. \tag{4.18}$$

Since $\|\nabla u\|_{L^2(\Gamma)} \leqq \delta\|\Delta u\|_{L^2(\Gamma)} + (c/\delta)\|u\|_{L^2(\Gamma)} \leqq \delta c + (c/\delta)\|u\|_{L^2(\Gamma)}$ for any $\delta \in (0, 1)$, the last inequality is deduced from the interior elliptic estimates and $\|\bar{u}\|_{L^2(B_1\times(-1,1))} = 1$.

Choosing δ so small that $\delta c \simeq \epsilon/2$, we obtain

$$\|u\|_{L^2(\Gamma)} \geqq c\delta\epsilon \geqq c\epsilon^2 \geqq 4^{-CN(t)}. \tag{4.19}$$

Finally we also have, by the elliptic interior estimate, that

$$\|u(\cdot t)\|_{L^2(B_{3/4}(0))} \leqq c. \tag{4.20}$$

Combining (4.19) and (4.20), we obtain a doubling condition (4.16) for $u^2(\cdot, t)$ on M. This completes the proof of Theorem 4.2.

Proof of Lemma 4.3: Let η be a smooth cutoff function such that $\eta = 1$ on $B_{5/8}(0)$, $\eta = 0$ outside $B_{1/4}(0)$, and $|\nabla\eta| \leqq C$. Let u_1 be the solution of

$$\begin{cases} Lu_1 = 0 & \text{in} \quad B_+ \\ u_1 = \eta u & \text{on} \quad \Gamma \\ u_1 = 0 & \text{on} \quad \partial B_+ \sim \Gamma \end{cases} \tag{4.21}$$

and let $u_2 = u - u_1$. Since $\|u\|_{H'(\Gamma)} \leqq \epsilon$, it is clear that

$$\|u_1\|_{L^2(B_{1/2})} \leqq c\epsilon, \quad \text{and} \quad \left\|\frac{\partial u_1}{\partial x_n}\right\|_{L^2(\Gamma)} \leqq c\epsilon. \tag{4.22}$$

Now we consider u_2. From (4.21) we have

$$Lu_2 = 0 \quad \text{in} \quad B_+, \quad u_2 = u - \eta u \quad \text{on} \quad \partial B_+. \tag{4.23}$$

Moreover,

$$\left\|\frac{\partial u_2}{\partial x_n}\right\|_{L^2(\Gamma)} \leqq \left\|\frac{\partial u}{\partial x_n}\right\|_{L^2(\Gamma)} + \left\|\frac{\partial u_1}{\partial x_n}\right\|_{L^2(\Gamma)} \leqq C\epsilon.$$

It suffices to show that $\|u_2\|_{L^2(B_{1/2}^+)} \leqq c\epsilon^\alpha$. This was done by a version of Carleman's inequality (see [14]).

Consider

$$u_2^*(x) = \begin{cases} u_2(x) & \text{for } x \in B_+ \\ 0 & \text{for } x \in B_- \end{cases},$$

and let η be as above; we apply Carleman's inequality to the function ηu_2^* to obtain, for all large $t < 0$, that

$$\|e^{t\phi(x_n)} \eta u_2^*\|_{L^2(B)} \leqq C_0 \|e^{t\phi(x_n)} L(\eta u_2^*)\|_{L^2(B)}. \tag{4.24}$$

Here $\phi(x_n) = (x_n + \frac{1}{8}) + \frac{1}{2}(x_n + \frac{1}{8})^2$.

As in [9], we obtain from (4.24) the following:

$$e^{tB} \|u_2^*\|_{L^2(B_{1/2})} \leqq C_0 \left(e^{tA} \left\| \frac{\partial(\eta u_2^*)}{\partial x_n} \right\|_{L^2(\Gamma)} + e^{tC} \|u_2\|_{L^2(B)} \right). \tag{4.25}$$

Here $A < B < C$ are positive constants. Hence

$$\|u_2\|_{L^2(B_{1/2}^+)} \leqq C_0 [e^{t(A-B)} c\epsilon + e^{t(C-B)}] \tag{4.26}$$

for all large negative t.

We choose t so that

$$\epsilon e^{t(A-B)} \simeq e^{t(C-B)},$$

i.e.,

$$\epsilon \simeq e^{t(C-A)}.$$

Then

$$\|u_2\|_{L^2(B_{1/2}^+)} \leqq c\epsilon^\alpha$$

follows from (4.26).

Remark (a). Let u be a nonzero solution of (1.11) in $B_1(0)$. Then we see, by Theorem 1.3, that

$$\int_{\partial B_r(a)} u^2 \, d\sigma \geqq 4^{-cN} \int_{\partial B_{2r}(a)} u^2 \, d\sigma$$

for $a \in B_{1-r_0}(0)$ and $r \in (0, r_0)$, where $N = N(a, r_0)$ and c is a positive constant which depends on λ, n, K_1, K_2.

Suppose now $u(a) = 0$ and $\Gamma_r(a)$ is the intersection of the ball $B_r(a)$ with a hyperplane passing through a. Then Lemma 4.3 implies that

$$\|\hat{u}\|_{H'(\Gamma_r)} \geqq c\|\hat{u}\|^{\alpha}_{L^2(B_r)}\|\hat{u}\|^{1-\alpha}_{L^2(B_{2r})}$$

for some positive constants c and α as in Lemma 2. Here

$$\|\hat{u}\|_{L^2(B_\rho)} = ⨍_{B_\rho} u^2\,dx = \frac{1}{\rho^n}\int_{B_\rho} u^2,$$

$$\|\hat{u}\|_{H'(\Gamma_\rho)} = \frac{1}{\rho^{n-1}}\int_{\Gamma_\rho} u^2\,d\alpha' + \frac{1}{\rho^{n-3}}\int_{\Gamma_\rho} |u_{x'}|^2\,dx'.$$

By Poincaré's inequality, one has also that

$$\int_{\Gamma_\rho(a)} u^2\,dx' \leqq c(n)\rho^2\int_{\Gamma_\rho(a)} |u_{x'}|^2\,dx'.$$

Therefore

$$\rho^{3-n}\int_{\Gamma_\rho(a)} |u_{x'}|^2\,dx' \geqq c\rho^{cN}$$

for $\rho \in (0, r_0)$.

Remark (b). As in [3], Theorem 4.2 remains true in the case that $\partial M \neq \emptyset$ and is real analytic. Also one can easily generalize Theorem 4.2 to the case that

$$\begin{cases} u_t = Lu & \text{in} \quad \Omega \times (0,\infty), \\ u(x,0) = u_0(x), & \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

where $\Omega \leqq \mathbb{R}^n$ is the bounded domain with real analytic boundary, and L is as in (1.11), where all coefficients are real analytic in $\bar{\Omega}$. If Ω and L are only suitably smooth, then a weak estimate as in [5] is valid.

Remark (c). It is easy to see from (4.7) that if $u_0(x) = \phi_j(x)$ for some j, then $N(t) \equiv \log(4\sqrt{\lambda_j} + \sinh 4\sqrt{\lambda_j})/(2\sqrt{\lambda_j} + \sinh 2\sqrt{\lambda_j}) \leqq \max(\log 2, 3\sqrt{\lambda_j})$. Theorem 4.2 implies, in particular, that

$$H^{n-1}\{x \in M : \phi_j(x) = 0\} \leqq c(n,g,M)\sqrt{\lambda_j}.$$

We should point out that in general it is possible to estimate $H^{n-1}\{x \in M : u(x,t) = 0\}$ in terms of

$$\lambda(t) = \frac{\int_M |\nabla u|^2(\cdot, t)\, dx}{\int_M u^2(\cdot t)\, dx}$$

for suitably large t (say $t \geqq 1$). One notices also that $\lambda(t)$ is a monotone nonincreasing function of t.

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