Russian Mathematical Surveys

SOME PROBLEMS OF THE QUALITATIVE THEORY OF SECOND ORDER ELLIPTIC EQUATIONS (CASE OF SEVERAL INDEPENDENT VARIABLES)

To cite this article: E M Landis 1963 *Russ. Math. Surv.* **18** 1

SOME PROBLEMS OF THE QUALITATIVE THEORY OF SECOND ORDER ELLIPTIC EQUATIONS (CASE OF SEVERAL INDEPENDENT VARIABLES)

by

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Contents

Introduction

This article is a continuation of the article "Some Problems of the Qualitative Theory of Elliptic and Parabolic Equations", published in Uspekhi Mat. Nauk, vol. XIV, part 1 (85), 1959. In it we considered the case of two independent variables.

According to the original plan the subsequent article should consider the same problems, but now for the case of several independent variables. However, circumstances so turned out that this subsequent article appears with a great delay. In the time which has elapsed, new success has been achieved in this field, and it appeared appropriate to alter somewhat the original plan: to restrict ourselves to the elliptic equations only, and instead, to include some new problems (on the behaviour of solutions of self-adjoint equations with non-smooth coefficients, the theorem concerning three spheres, and some others), and to devote another article to the parabolic equations.

In this article, we consider exclusively the second order linear elliptic equation

$$\sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0. \qquad (1)$$

It is always supposed here, that the sign of $c(x)$ is taken so that the maximum principle is fulfilled.

Chapter I

PROPERTIES OF SOLUTIONS WITH CONSTANT SIGN

§0. Introduction

This chapter concerns properties of those solutions of second order linear elliptic equations, which preserve a constant sign in the domain in which they are determined. We may assume that this sign is positive.

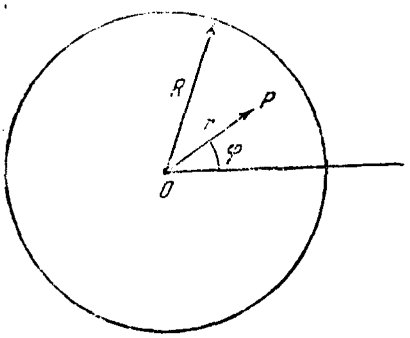

Fig. 1.

One of the principal facts, characterizing the behaviour of positive harmonic functions, is Harnack's inequality: let a positive harmonic function u be defined in a circle of radius R, then (fig. 1) at any point $P(r, \varphi)$ we have the inequality

$$\frac{R-r}{R+r} < \frac{u(P)}{u(O)} < \frac{R+r}{R-r},$$

where r is the distance from P to the centre O of the circle.

This inequality shows that far away from the boundary of the domain, a positive harmonic function changes slowly.

For solutions of the elliptic equation (1) analogous theorems have the following forms:

I. In the case of two space variables, Serrin [1] proved the following theorem. Let equation (1) be uniformly elliptic in the circle of radius $R < 1$, i.e.

$$0 < a < \frac{\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k}{\sum_{i=1}^{n} \xi_i^2} < A. \tag{0.1}$$

Let the remaining coefficients be bounded by a constant A, and let $c \leqslant 0$. Then, for any $r < R$ there exist positive constants C_1 and C_2, depending on a, A and r/R, such that in the circle of radius r

$$C_1 < \frac{u(P)}{u(0)} < C_2. \tag{0.2}$$

2. For the case of several independent variables the same author [1] proved the following theorem. Let equation (1) be defined in a sphere of radius $R < 1$, let all the previous conditions be fulfilled, and besides, let the coefficients a_{ik} be continuous at points on the surface of the sphere and satisfy a Dini condition there: for any pair of points P and Q, where P is a point of the sphere, and Q is a point of the surface of the sphere,

$$|a_{ik}(P) - a_{ik}(Q)| < \varphi(|PQ|), \tag{0.3}$$

where $\int_0 \frac{\varphi(s)}{s}\,ds < \infty$. Let $\int_0^{2R} \frac{\varphi(s)}{s}\,ds = \Delta$. Then, as before, the inequality (0.2) holds for any $r < R$, but C_1 and C_2 depend not only on a, A and r/R, but also on Δ.

We note that although very little is demanded here of the coefficients, (in all, only the Dini condition, and then only on the surface on the sphere), however, between the theorems of Serrin in the case of two and of several variables, there is an essential difference in that in the first case the theorem is generalized automatically to the case of the quasi-linear equation

$$\sum_{i,k=1}^{n} a_{ik}\left(x, u, \frac{\partial u}{\partial x_j}, \frac{\partial^2 u}{\partial x_r \partial x_s}\right) \frac{\partial^2 u}{\partial x_i \partial x_k} + \ldots = 0,$$

but in the second case we have not the possibility of going beyond the limits of linearity in such a simple way. Such a possibility appeared after Nash [31] and Giorgi [3] proved, that the solution of a self-adjoint elliptic equation satisfies a Hölder condition, independently of the smoothness of the coefficients (a simple proof of this theorem was given by Moser [4]). Further, Kruzhkov [32], [33] gave a method, allowing the transfer of the interior estimates for the solution of the elliptic equation

$$\sum_{i,k} \frac{\partial}{\partial x_i}\left(a_{ik}(x) \frac{\partial u}{\partial x_k}\right) + c(x)u = f(x)$$

to solutions of the equation

$$\sum_{i,k} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + \sum_i b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x), \tag{0.4}$$

in particular, the estimates of Nash and Giorgi. This method consists in considering the solution $u(x_1, \ldots, x_n)$ of equation (0.4) as the solution $u(x_1, \ldots, x_n, y) \equiv u(x_1, \ldots, x_n)$ of the equation

$$\sum_{i,k} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + \sum_i \frac{\partial}{\partial y}\left(b_i(x)\, y\,\frac{\partial u}{\partial x_i}\right) + \sum_i \frac{\partial}{\partial x_i}\left(b_i(x)\, y\,\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right) + c(x)u = f(x),$$

where the constant K is selected so large that the form $\Sigma a_{ik}\xi_i\xi_k + \Sigma b_i\xi_i\eta + K\eta^2$ is made positive definite.

By an analogous method one can include lower terms in a non-self-adjoint equation, and also terms of the form $cu + f$ in an equation containing terms with derivatives only.

Finally, in 1961 Moser [34] proved that Harnack's inequality holds for the self-adjoint equation

$$\sum_{i,k} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) = 0,$$

$$a_{ik} = a_{ki}, \qquad 0 < a \leqslant \frac{\sum a_{ik}\xi_i\xi_k}{\sum \xi_i^2} \leqslant A$$

with arbitrary measurable coefficients (the equation is understood in the sense of an integral identity).

The method of Kruzhkov mentioned above, permits the introduction into this equation of lower terms with arbitrary bounded coefficients.

In the case of harmonic functions an immediate consequence of Harnack's inequality is Liouville's theorem: a positive harmonic function, defined in the whole plane is a constant. In the general case it will be the same, if the constant is a solution of the equation, and if Harnack's inequality (0.2) is satisfied for a sphere of any radius, the constants C_1 and C_2 depending, for any R, only on the ratio r/R. Hence, for the equation

$$\sum_{i,k=1}^{n} a_{ik}\frac{\partial^2 u}{\partial x_i\,\partial x_k} = 0 \tag{0.5}$$

when $n = 2$, Liouville's theorem holds for any coefficients, if only the inequality (0.1) be fulfilled uniformly (a theorem by Serrin [1]: a positive solution of equation (0.5) for $n = 2$, defined in the whole plane, is a constant).

For $n > 2$, this was proved under the hypothesis that the coefficients of equation (0.4) satisfy Dini's condition at infinity (theorem of Gilbarg and Serrin [2]).

A few words concerning Liouville's theorem for the elliptic equation (0.4), without the hypothesis of uniform ellipticity.

As is known from an example by Bernstein [5], Liouville's theorem, in the formulation which demands boundedness of the solution on one side, is not true in this case. If one demands boundedness of the solution on the two sides, such a Liouville theorem, in the case of two variables, was proved by Bernstein [5]. Adel'son-Vel'skii proved [6], that in this case, Liouville's theorem is a fact connected with the geometry of the graph of the function. In fact, he proved the following theorem. Let $f(x, y)$ be a continuous function, defined in the whole xy-plane. Let the graph of this function be such that it is impossible to cut off a "cap" by any plane whatever (i.e. for any linear function $ax + by + c$, the set of points xy, where $f(x, y) > ax + by + c$ or $f(x, y) < ax + by + c$ has not a bounded component). Then, either $f(x, y)$ grows at infinity not slower than linearly, or the graph of $f(x, y)$ is a cylinder with generator parallel to the plane of xy.

An example by Hopf [7] proves that, in the case of three independent variables, this is indeed not true, even for the solution of the elliptic equation (0.4) (without the condition of uniform ellipticity). For uniformly elliptic equations, the two-sided theorem of Liouville was proved by Gilbarg and Serrin [2] under the hypothesis that the coefficients in the equation have limits at infinity.

We return to Harnack's inequality. For an unbounded domain, not coinciding with the whole space, we obtain from it an upper estimate for the rate of growth and decay of a positive solution in dependence on the "width" of the domain. Thus, if the solution of equation (1), for which Harnack's inequality is correct, is positive in the cylinder $\sum_{i=2}^{n} x_i^2 < h^2$, $x_1 > 0$, then, in the narrower cylinder $\sum_{i=2}^{n} x_i^2 < h_1^2 < h^2$, this solution grows and decays not faster than the exponential:

$$\frac{1}{C} e^{-Mx_1} < u(x_1, \ldots, x_n) < Ce^{Mx_1}.$$

The constant M depends on h, h_1/h and on the constants in Harnack's inequality (0.2).

The solution, determined inside a cone, will, in a narrower cone, on going to infinity or on approaching the vertex of the cone, increase and decrease not faster than according to a power, the index being proportional to the angle at the vertex.

Estimates of a similar kind are obtained at once by applying Harnack's inequality to a sequence of spheres, lying in the domain, and such that the centre of each sphere lies inside the previous one, and is situated inside a sphere concentric with this previous one, and having its radius smaller in a given constant ratio (for example, one half). We shall not, therefore, dwell on them in more detail.

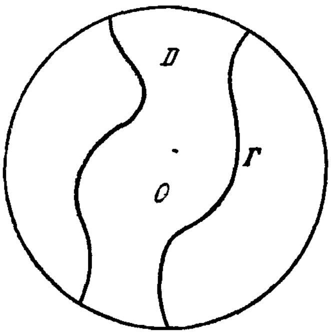

Fig. 2.

For positive solutions, vanishing on the boundary of the domain (if the domain is infinite) or on part of the boundary, one can obtain an estimate on the other side: the narrower the domain, the quicker the solution increases or decreases in it. Corresponding estimates follow from such a fact. Let a domain D, situated inside a sphere of radius $R < 1$ (fig. 2), contain the centre O of the sphere, and have limit points on its periphery. We denote by Γ that part of the boundary of the domain, which is strictly inside the sphere. Let the solution $u(x)$ of equation (1) be defined in D, be positive in D, and vanish on Γ. One must make the assumptions with respect to the coefficients as in the paper by Serrin mentioned (but to demand that Dini's condition be satisfied, now not on the boundary, but at all points of space).

Then, because the ratio of the measure of the domain D to the volume of the sphere is smaller than some constant $\varepsilon > 0$, depending on α, A, Δ (the number Δ is defined by the Dini condition), it follows that

$$\sup_{x\in D} u(x) > 2u(0).$$

For a self-adjoint equation (as in the case of Giorgi) a similar statement can be proved under the assumption of measurability only of the coefficients.

This statement makes it possible to obtain a theorem of Phragmén-Lindelöf type [30], and also to obtain a lower estimate for the rate of growth and decay of the solution in the neighbourhood of a boundary point, depending on the structure of the domain in the neighbourhood of this point.

In the first chapter the principal place is taken by the establishment of this fact of the growth of the solution in a narrow domain (lemmas 3.1 and 7.1), which is then applied to estimates of the growth or decay of the solution in domains of different form.

§1. Conditions imposed on the coefficients of the equation. Some notations

In this section we consider the equation

$$Lu \equiv \sum_{i,k=1} a_{ik}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_k} + \sum_{i=1} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0. \tag{1.0}$$

Concerning the coefficients, we shall suppose that they are measurable, have their moduli bounded by unity, and satisfy the inequalities

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \geqslant \alpha\sum_{i=1}^{n}\xi_i^2, \qquad \alpha > 0, \tag{1.1}$$

and

$$c(x) \leqslant 0. \tag{1.2}$$

That we bound the coefficients by unity, and not by some other constant, clearly plays no part whatever, since we can divide the entire equation by this constant, altering correspondingly inequality (1.1).

Besides, in §§3-6 of this chapter, we suppose that the coefficients a_{ik} are all continuous, and have the common modulus of continuity $\omega(r)$, satisfying the condition

$$\int_0^d \frac{\omega(r)\,dr}{r} = \Delta < \infty,$$

where d is the diameter of the domain in which the equation is defined (d may be infinite).

We denote equation (1.0) *together with these conditions by* (1.0.A).

In future Q_R will denote the n-dimensional sphere with centre at the origin of coordinates O, and radius R.

By $\mu_k E$ we denote the k-dimensional Hausdorff measure of a set E in n-dimensional space. In particular, $\mu_n E$ is the Lebesgue n-dimensional measure of the set E.

§2. Continuation of a quadratic form from a domain to the whole space.

In this section we consider the following problem. There is given a domain $D \subset R_n$. In this domain the coefficients of a quadratic form are given

$$\sum_{i,k=1}^{n} a_{ik}(x)\,\xi_i\xi_k. \tag{2.0}$$

Let each of the coefficients $a_{ik}(x)$ be uniformly continuous in D, the quadratic form satisfying the inequalities

$$|a_{ik}(x)| \leqslant 1, \quad \sum_{i,k=1}^{n} a_{ik}(x)\,\xi_i\xi_k \geqslant \alpha \sum_{i=1}^{n} \xi_i^2, \quad x \in D. \tag{2.1}$$

Is it possible to continue the coefficients of the quadratic form to the whole space, so that the modulus of continuity of each of the coefficients in increased by not more than a constant multiplicative factor (depending only on the dimensionality of space), and so that the inequality (2.1) is preserved?

This problem has a positive solution. To obtain it, we consider the following algorithm of continuation of a function from D to the whole space.

Let $f(x)$ be a function uniformly continuous in D, and its modulus of continuity $\omega(r)$ be a convex function[1]

$$\omega(r_1 + r_2) \leqslant \omega(r_1) + \omega(r_2), \quad r_1,\ r_2 > 0.$$

We continue $f(x)$ by continuity in $\overline{D}$, and let x be a point of R_n, not belonging to D.

We denote the intersection of $\overline{D}$ with a sphere of radius r and centre at the point x by $D_x(r)$, the n-dimensional measure of $D_x(r)$ by $m_x(r)$, and the distance from x to the domain D by ρ_x. We put

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[1] The assumption of convexity of $\omega(r)$ is essential so that the function can be continued with increase of modulus of continuity by a factor not depending on the domain.

$$\tilde{f}(x) = \frac{\int\limits_{\rho_x}^{\infty} \left[\frac{\int\limits_{D_x(r)} f(y)\,dy}{r^3 m_x(r)} \right] dr}{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^3}}. \tag{2.2}$$

Let x_0 be any point belonging to $\overline{D}$, and $|x - x_0| = \delta$. We prove that

$$|\tilde{f}(x) - f(x_0)| < 4\omega(\delta). \tag{2.3}$$

We have

$$\tilde{f}(x) - f(x_0) = \frac{\int\limits_{\rho_x}^{\infty} \left[\frac{\int\limits_{D_x(r)} f(y)\,dy}{r^3 m_x(r)} \right] dr}{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^3}} - f(x_0) =$$

$$= \frac{\int\limits_{\rho_x}^{\infty} \left[\frac{\int\limits_{D_x(r)} f(y)\,dy}{r^3 m_x(r)} \right] dr}{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^3}} - \frac{\int\limits_{\rho_x}^{\infty} \left[\frac{\int\limits_{D_x(r)} f(x_0)\,dy}{r^3 m_x(r)} \right] dr}{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^3}}$$

or

$$|\tilde{f}(x) - f(x_0)| \leqslant \frac{\int\limits_{\rho_x}^{\infty} \left[\frac{\int\limits_{D_x(r)} |f(y) - f(x_0)|\,dy}{r^3 m_x(r)} \right] dr}{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^3}}. \tag{2.4}$$

From the convexity of the function $\omega(r)$ it follows that

$$|f(x) - f(x_0)| \leqslant \omega(r + \delta) \leqslant \omega(\delta) \left\{ \left[\frac{r}{\delta} \right] + 1 \right\} + \omega(\delta),$$

where $r = |x - y|$, or

$$|f(y) - f(x_0)| \leqslant \omega(\delta) \frac{r}{\delta} + 2\omega(\delta). \tag{2.5}$$

From (2.4) and (2.5) we get

$$|\tilde{f}(x) - f(x_0)| \leqslant \frac{\omega(\delta)}{\delta} \frac{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^2}}{\int\limits_{\rho_x}^{\infty} \frac{dr}{r^3}} + 2\omega(\delta) = \omega(\delta) \frac{\rho_x}{\delta} + 2\omega(\delta),$$

and since $\rho_x \leqslant \delta$, then

$$|\tilde{f}(x) - f(x_0)| < 4\omega(\delta).$$

We denote now by Q^x the sphere of radius ρ_x with centre at the point x, and we complete the definition of the function f at the point $x \notin \overline{D}$ as follows:

$$f(x) = \frac{\int\limits_{Q^x} \tilde{f}(y)\,dy}{\mu_n Q^x}. \tag{2.6}$$

We shall prove that

$$|f(x_2) - f(x_1)| \leqslant C\omega(|x_2 - x_1|),$$

where $C = 160n$, for all pairs of points x_1, $x_2 \in R_n$.

Consider first the case when $x_1 \in \overline{D}$ and $x_2 \in R_n \setminus \overline{D}$. We have

$$|f(x_2) - f(x_1)| = \left| \frac{\int\limits_{Q^{x_2}} \tilde{f}(y)\,dy}{\mu_n Q^{x_2}} - f(x_1) \right| \leqslant \frac{\int\limits_{Q^{x_2}} |\tilde{f}(y) - f(x_1)|\,dy}{\mu_n Q^{x_2}};$$

since $|y - x_2| \leqslant \rho_{x_2} \leqslant |x_2 - x_1|$, then

$$|y - x_1| \leqslant 2|x_2 - x_1|,$$

and from the inequality (2.3) we get

$$|\tilde{f}(y) - f(x_1)| \leqslant 4\omega(2|x_2 - x_1|) \leqslant 8\omega(|x_2 - x_1|),$$

i.e.

$$|f(x_2) - f(x_1)| \leqslant \frac{\int\limits_{Q^x} 6\omega(|x_2 - x_1|)\,dy}{\mu_n Q^x} = 8\omega(|x_2 - x_1|). \tag{2.7}$$

We now turn to the case where both x_1 and x_2 are outside $\overline{D}$.

We denote $|x_2 - x_1| = \delta$, and let $\rho_{x_1} \geqslant \rho_{x_2}$. If $\rho_{x_1} \leqslant \delta$, then, denoting by x_0 the point of $\overline{D}$ nearest to x_1, we have

$$|x_1 - x_0| \leqslant \delta, \quad |x_2 - x_0| \leqslant 2\delta,$$

and in view of inequality (2.7)

$$f(x_2) - f(x_1)| \leqslant |f(x_2) - f(x_0)| + |f(x_1) - f(x_0)| \leqslant \\ \leqslant 8\omega(2\delta) + 8\omega(\delta) = 24\omega(\delta). \tag{2.8}$$

If, however, $\rho_{x_1} > \delta$, then, denoting as before by x_0 the point of $\overline{D}$ nearest to x_1, we find that for any point $y \in Q^{x_1} \cup Q^{x_2}$

$$|y - x_0| < 4\varrho_{x_1},$$

and in view of (2.3), for any such point y one has

$$|\tilde{f}(y) - f(x_0)| < 4\omega(4\varrho_{x_1}) \leqslant 16\omega(\varrho_{x_1}).$$

We put $\varphi(y) = \tilde{f}(y) - f(x_0)$. We have

$$|\varphi(y)| \leqslant 16\omega(\varrho_{x_1}) \quad \text{if} \quad y \in Q^{x_1} \cup Q^{x_2}. \tag{2.9}$$

Then

$$f(x_2)-f(x_1)=[f(x_2)-f(x_0)]-[f(x_1)-f(x_0)]=$$

$$=\frac{\int\limits_{Q^{x_2}}[\widetilde{f}(y)-f(x_0)]\,dy}{\mu_n Q^{x_2}}-\frac{\int\limits_{Q^{x_1}}[\widetilde{f}(y)-f(x_0)]\,dy}{\mu_n Q^{x_1}}=\frac{\int\limits_{Q^{x_2}}\varphi(y)\,dy}{\mu_n Q^{x_2}}-\frac{\int\limits_{Q^{x_1}}\varphi(y)\,dy}{\mu_n Q^{x_1}}=$$

$$=\frac{\int\limits_{Q^{x_2}}\varphi(y)\,dy-\int\limits_{Q^{x_1}}\varphi(y)\,dy}{\mu_n Q^{x_1}}+\left(\frac{1}{\mu_n Q^{x_2}}-\frac{1}{\mu_n Q^{x_1}}\right)\int\limits_{Q^{x_2}}\varphi(y)\,dy=$$

$$=\frac{\int\limits_{Q^{x_2}\setminus Q^{x_1}}\varphi(y)\,dy}{\mu_n Q^{x_1}}-\frac{\int\limits_{Q^{x_1}\setminus Q^{x_2}}\varphi(y)\,dy}{\mu_n Q^{x_1}}+\left(\frac{1}{\mu_n Q^{x_2}}-\frac{1}{\mu_n Q^{x_1}}\right)\int\limits_{Q^{x_2}}\varphi(y)\,dy.$$

We estimate the modulus of each of the three terms on the right-hand side of the last equation

$$\left|\frac{\int\limits_{Q^{x_2}\setminus Q^{x_1}}\varphi(y)\,dy}{\mu_n Q^{x_1}}\right|\leqslant\frac{16\omega(\varrho_{x_1})\,\mu_n(Q^{x_2}\setminus Q^{x_1})}{\mu_n Q^{x_1}}\leqslant\frac{16\omega(\varrho_{x_1})\,\sigma_n\varrho_{x_1}^{n-1}2\delta}{\omega_n\varrho_{x_1}^n}\leqslant\frac{32n\omega(\varrho_{x_1})\,\delta}{\varrho_{x_1}}. \tag{2.10}$$

Here ω_n and σ_n are respectively the volume and area of the surface of the unit n-dimensional sphere. The second term is estimated similarly:

$$\left|\frac{\int\limits_{Q^{x_1}\setminus Q^{x_2}}\varphi(y)\,dy}{\mu_n Q^{x_1}}\right|\leqslant\frac{32n\omega(\varrho_{x_1})\,\delta}{\varrho_{x_1}}. \tag{2.11}$$

Finally

$$\left|\left(\frac{1}{\mu_n Q^{x_1}}-\frac{1}{\mu_n Q^{x_1}}\right)\int\limits_{Q^{x_2}}\varphi(y)\,dy\right|\leqslant\left(\frac{1}{\omega_n\varrho_{x_2}^n}-\frac{1}{\omega_n\varrho_{x_1}^n}\right)16\omega(\varrho_{x_1})\,\omega_n\varrho_{x_2}^n\leqslant$$

$$\leqslant 16\,\frac{\varrho_{x_1}^n-\varrho_{x_2}^n}{\varrho_{x_1}^n}\,\omega(\varrho_{x_1})\leqslant\frac{16n\omega(\varrho_{x_1})\,\delta}{\varrho_{x_1}}. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12), we get

$$|f(x_2)-f(x_1)|\leqslant\frac{80n\omega(\varrho_{x_1})\,\delta}{\varrho_{x_1}}\leqslant\frac{80n\omega(\delta)\left(\left[\frac{\varrho_{x_1}}{\delta}\right]+1\right)\delta}{\varrho_{x_1}}<160n\omega(\delta). \tag{2.13}$$

In conjunction with (2.7) and (2.8) this shows that for all x_1 and x_2 belonging to R_n, we have the inequality

$$|f(x_2)-f(x_1)|<160n\omega(|x_2-x_1|). \tag{2.14}$$

Our algorithm of continuation of a function assigns to any function $f(x)$, uniformly continuous in D, a function $F(x)$, defined in R_n, and coinciding with $f(x)$ in D. We denote by A the operator carrying f into F:

$$F=Af.$$

From (2.2) and (2.6) it follows that the operator is *linear*, and

$$\inf_{x\in D} f(x)\leqslant Af\leqslant\sup_{x\in D} f(x). \tag{2.15}$$

We can now concern ourselves with the continuation of the quadratic form.

LEMMA 2.1. *Let the coefficients a_{ik} of a quadratic form (2.0) be defined in a domain D. Let the coefficients a_{ik} $(i, k = 1, \ldots, n)$ be uniformly continuous in D, with a convex modulus of continuity $\omega_{ik}(r)$, and the quadratic form satisfy the inequality (2.1). Then it is possible to continue the coefficients of the quadratic form to the whole space so that the modulus of continuity $\Omega_{ik}(r)$ of the coefficient $A_{ik}(x)$, the continuation of a_{ik}, satisfies the inequality*

$$\Omega_{ik}(r) \leqslant 160n\omega_{ik}(k) \tag{2.16}$$

and for the continued quadratic form we have the inequalities

$$|A_{ik}| \leqslant 1, \quad \sum_{i,k=1}^{n} A_{ik}(x)\xi_i\xi_k \geqslant \alpha \sum_{i=1}^{n} \xi_i^2, \quad x \in R_n. \tag{2.17}$$

PROOF. We put $A_{ik}(x) = Aa_{ik}(x)$, where A is the operator of continuation previously constructed. The inequality (2.16) is fulfilled in consequence of (2.14). Because of the linearity of the operator A

$$\sum_{i,k=1}^{n} A_{ik}(x)\xi_i\xi_k = \sum_{i,k=1}^{n} \xi_i\xi_k Aa_{ik}(x) = A\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k.$$

Applying inequality (2.15), we get

$$\inf_{x\in R_n} \frac{\sum_{i,k=1}^{n} A_{ik}(x)\xi_i\xi_k}{\sum_{i=1}^{n}\xi_i^2} \geqslant \inf_{x\in D} \frac{\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k}{\sum_{i=1}^{n}\xi_i^2} \geqslant \alpha,$$

the inequality $|A_{ik}| \leqslant 1$ follows immediately from (2.15), and the lemma is proved.

If now we have the equation (1.0), whose coefficients a_{ik} are uniformly continuous in D, then this lemma, evidently, permits the continuation of this equation to the whole space, so that the inequalities (1.1), (1.2) are preserved, and the modulus of continuity of the coefficients increases not more than $160n$ times. It is here assumed that the moduli of continuity are convex functions.

§3. The principal lemma

In this section the following restrictions will be placed on the coefficients of equation (1.0).

The coefficients a_{ik} $(i, k = 1, \ldots, n)$ have a common modulus of continuity $\omega(r)$, which is a convex function, and satisfies the condition

$$\int_0^d \frac{\omega(r)}{r}\,dr = \Delta < \infty, \tag{3.1}$$

where d is the diameter of the domain D, in which the equation is given.

We shall use the following result of Serrin [1]. Let the equation (1.0.A) be defined in the sphere Q_R, $R \leqslant 1$. Let S_R be the surface of the sphere Q_R. Then, there exists a function $K(x, x')$, $x \in Q_R$, $x' \in S_R$, such that:

1) for any function $\varphi(x')$, defined and continuous in S_R, the function

$$v(x) = \frac{1}{\mu_{n-1} S_R} \int_{S_R} K(x, x') \varphi(x') \, d\sigma$$

satisfies the condition

$$Lv \leqslant 0; \tag{3.2}$$

2) $$\lim_{x \to x'} v(x) = \varphi(x');$$

3) $$0 < K(x, x') < A, \quad x' \in S_R, \tag{3.3}$$

where A is a constant depending on the constant a of the inequality (1.1), on the constant Δ of the inequality (3.1), and on the dimensionality n of the space.

We now prove the following lemma.

LEMMA 3.1 (principal lemma). *Let a domain D be situated in Q_R, $R \leqslant 1$, contain the centre O of the sphere, and have limit points on the boundary of the sphere S_R. Let Γ be that part of the boundary of the domain, which lies strictly inside the sphere Q_R.*

There exists a constant M, depending only on the constant α of inequality (1.1), the constant Δ of inequality (3.1) and on the dimensionality n of space, so that if

$$\mu_n D < \frac{\mu_{n-1} Q_R}{M} \tag{3.4}$$

and in D the equation (1.0.A) is given, then any positive solution of it $u(x)$, given in D, continuous in $\bar{D}$, and vanishing on Γ, satisfies the inequality

$$u(0) < \frac{1}{2} \max_{x \in D} u(x). \tag{3.5}$$

PROOF. From the inequality (3.4), it follows that there exists a number r, $0 < r < R$, such that, if Q_r denotes the sphere with centre at the point O and radius r, and S_r is the boundary of this sphere and Γ_r is the intersection $S_r D$, then

$$\mu_{n-1} \Gamma_r < \frac{1}{M} \mu_{n-1} S_r. \tag{3.6}$$

We now, according to lemma 2.1, continue the equation (1.0.A) from the domain $D_r = Q_r \cap D$ to the sphere Q_r, so that inequalities (1.1) and (1.2) are preserved, and the modulus of continuity of the coefficients is increased by a factor not larger than $160n$. Then, $\Omega(r)$ denoting the modulus of continuity of the continued coefficients, we get

$$\int_0^d \frac{\Omega(r)}{r} dr < 160 n \Delta.$$

We put

$$\varphi(x') = \begin{cases} u(x') & \text{if } x' \in \Gamma_r, \\ 0 & \text{if } x' \in S_r \setminus \Gamma_r. \end{cases}$$

Further, we put

$$v(x) = \frac{1}{\mu_{n-1} S_r} \int_{S_r} K(x, x') \varphi(x') \, d\sigma,$$

where $K(x, x')$ is the kernel constructed for the sphere Q_r and the continued equation, and having the properties 1), 2), 3).

The boundary of D_r consists of Γ_r and points belonging to Γ. On Γ_r we have $v(x) = u(x)$, and at points belonging to Γ, $u = 0$, $v(x) > 0$, and hence $v(x) \geqslant u(x)$. Hence, the inequality $v(x) > u(x)$ is satisfied everywhere on the boundary of D_r. Whence, from (3.2), it follows that

$$u(x) \leqslant v(x) \quad \text{in} \quad D_r.$$

Hence

$$u(0) \leqslant v(0) = \frac{1}{\mu_{n-1} S_r} \int_{S_r} K(0, x') \varphi(x') \, d\sigma.$$

Then, because of the inequality (3.3)

$$\frac{1}{\mu_{n-1} S_r} \int_{S_r} K(0, x') \varphi(x') \, d\sigma < \frac{A}{M} \max_{x' \in \Gamma_r} u(x') \leqslant \frac{A}{M} \max_{x \in \bar{D}} u(x)$$

and, putting $M = 2A$, we arrive at (3.5).

In the paper [8] (p. 24), we constructed an example, showing that the restriction placed on the size of the radius R of the sphere Q_R is essential for the correctness of lemma 3.1 for $n = 2$. Of course, such an example is easily constructed for any n.

To ensure that the lemma remains true for any R (and the constant M does not depend on R), it is necessary to place restrictions on the coefficients b_i. The simplest of all is to put $b_i \equiv 0$; here we shall assume this. Further, we put $c \equiv 0$, and consider the equation

$$\sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_k} = 0. \tag{3.7}$$

The transformation

$$x_i' = \frac{x_i}{R} \qquad (i = 1, \ldots, n)$$

reduces this equation to the equation

$$\sum_{i,k=1}^{n} a_{ik}(Rx') \frac{\partial^2 u}{\partial x_i' \, \partial x_k'} \equiv \sum_{i,k=1}^{n} A_{ik}(x') \frac{\partial^2 u}{\partial x' \, \partial x_k'} = 0$$

with the same constant α in the inequality (1.1). This transformation changes the sphere Q_R into a sphere of radius 1, and preserves the ratio

$$\frac{\mu_n D}{\mu_n Q_R}.$$

Unfortunately, this transformation *alters the modulus of continuity*. In order that the inequality (3.1) should be satisfied for the transformed equation with the constant Δ, not depending on R, we require that for the original equation there should be fulfilled the inequality

$$\int_0^{2R} \frac{\omega(r)}{r}\, dr < \frac{\Delta}{R}\,. \tag{3.8}$$

We denote equation (3.7), *together with condition* (3.8), *by* (1.0.B).

For such an equation we obtain the lemma.

LEMMA 3.2. *Let Q_R be a sphere of arbitrary radius R, with centre at the point O. Let D be a domain, containing the centre of the sphere, and having limit points on the boundary of the sphere. Let Γ be that part of the boundary of the domain D, which lies strictly inside the sphere Q_R.*

Let equation (1.0.B) *be given in D, and a positive solution u(x) of it be known, continuous in $\overline{D}$, and vanishing on Γ.*

There exists a constant M, depending only on the constant α of the inequality (1.1) *and on the constant Δ of the inequality* (3.8), *such that from*

$$\mu_n D < \frac{\mu_n Q_R}{M}\,,$$

it follows that

$$u(0) < \frac{1}{2} \max_{x \in \overline{D}} u(x).$$

The problem of the nature of the conditions, which must be placed on the coefficients b_i so that the lemma should remain true, was investigated by A.A. Novruzov [9]. I shall not dwell here on these conditions. I only note that the principal of these is the following: We represent equation (1.0) in the form

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik} \frac{\partial u}{\partial x_k}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0,$$

it is then necessary that the sum $\sum_{i=1}^{n} b_i x_i$ should be positive.

The presence of R in the denominator of the right-hand side of inequality (3.8) is an unfortunate limitation. If the coefficients a_{ik} are differentiable, then this limitation leads to the demand that the derivatives decrease inversely proportional to R. Evidently, this demand is not necessary, and is connected only with the method of proof. There are grounds for believing that for the correctness of lemma (3.2), it is not necessary to demand anything of the coefficients other than the inequality (1.1). It would be interesting to establish this fact.

In the case when the equation has the self-adjoint form

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik} \frac{\partial u}{\partial x_k}\right) = 0,$$

this is actually so: the coefficients a_{ik} may be arbitrary measurable functions (the function u here satisfies the equation in the sense of an integral identity). If the coefficients satisfy inequality (1.1), then lemma 3.2 is correct. This problem will be considered in §§7-8.

§4. Character of growth of a solution in a bounded domain

THEOREM 4.1. *Let Q_R be a sphere of radius $R \leqslant 1$ with centre at the point O. Let D be a domain, situated inside the sphere, containing the point O, and having limit points on the boundary of the sphere. Let Γ be that part of the boundary of the domain D, which is situated strictly inside Q_R. Let equation (1.0.A) be given in D. Let, also,*

$$\mu_n D = \sigma < \frac{\mu_n Q_R}{M}, \tag{4.1}$$

where M is the constant of lemma 3.1, and let a positive solution u be determined in D, continuous in $\bar{D}$, and vanishing on Γ. Then

$$u(0) \leqslant e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \max_{x \in \bar{D}} u(x), \tag{4.2}$$

where C is a constant depending on the constant α of inequality (1.1), the constant Δ of inequality (3.1), and on the dimensionality n of space.

PROOF. We consider first the case when $\sigma < \frac{\mu_n Q_R}{2^{4n-4}M}$. We put

$$N = \left[\left(\frac{\omega_n R^n}{4M\sigma} \right)^{\frac{1}{n-1}} \right],$$

where ω_n is the volume of the unit n-dimensional sphere.

We denote by C_k ($k = 1, 2, \ldots, N-1$) the sphere $|x| = \frac{kR}{N}$. We put

$$m_k = \max_{x \in C_k \cap D} u(x) \quad (k = 1, 2, \ldots, N-1).$$

Let these maxima be attained respectively at the points $x^1, \ldots, x^{N-1}$. We put $x^0 = 0$, and $m_0 = u(0)$.

We denote by Q^k the sphere of radius R/N, with centre at the point x^k ($k = 0, 1, \ldots, N-1$). Let

$$g_k = Q^k \cap D.$$

Let, further, $k_1, \ldots, k_s$ be those of the numbers $0, 1, \ldots, N-1$, arranged in increasing order of magnitude, for which

$$\mu_n g_k < \frac{\omega_n \left(\frac{R}{N} \right)^n}{M}. \tag{4.3}$$

The number s of these integers, in view of the inequality

$$\sum_{k=0}^{N-1} \mu_n g_k \leqslant \sigma \tag{4.4}$$

is not less than $\frac{1}{2}N$. In fact, in the contrary case, the number of different values of k for which one has the inequality

$$\mu_n g_k \geqslant \frac{\omega_n \left(\frac{R}{N}\right)^n}{M},$$

is not less than

$$\frac{N}{2} - 1 \geqslant \frac{N}{4}.$$

Whence

$$\sum_{k=0}^{N-1} \mu_n g_k > \frac{N}{4} \frac{\omega_n \left(\frac{R}{N}\right)^n}{M} = \frac{\omega_n R^n}{4MN^{n-1}} \geqslant \frac{\omega_n R^n 4M\sigma}{4M\omega_n R^n} = \sigma,$$

and we arrive at a contradiction to inequality (4.4).

By the inequality (4.3), for each i ($i = 1, \ldots, s$), there holds, on the basis of lemma 3.1, the inequality

$$m_{k_i} < \frac{1}{2} \max_{x \in \bar{g}_{k_i}} u(x),$$

and since, by the maximum principle,

$$\max_{x \in \bar{g}_{k_i}} u(x) \leqslant m_{k_{i+1}},$$

then

$$u(0) = m_0 < 2^{-s} \max u(x) \leqslant 2^{-\frac{N}{2}} \max_{x \in \bar{D}} u(x),$$

or

$$u(0) < 2^{-\frac{1}{2}\left[\left(\frac{\omega_n R^n}{4M\sigma}\right)^{\frac{1}{n-1}}\right]} \max_{x \in \bar{D}} u(x) \leqslant e^{-\frac{\omega_n^{\frac{1}{n-1}} \ln 2}{16M^{\frac{1}{n-1}}} \frac{R^{\frac{1}{n-1}}}{\sigma^{\frac{1}{n-1}}}} \max_{x \in \bar{D}} u(x).$$

Putting

$$\frac{16M^{\frac{1}{n-1}}}{\omega_n^{\frac{1}{n-1}} \ln 2} = C,$$

we arrive at the inequality

$$u(0) \leqslant e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \max_{x \in \bar{D}} u(x).$$

There remains for us to consider the case when

$$\sigma \geqslant \frac{\mu_n Q_R}{2^{4n-4}M} = \frac{\omega_n R^n}{2^{4n-4}M}. \tag{4.5}$$

By the inequality (4.1), according to lemma 3.1, we obtain

$$u(0) < 2^{-1} \max_{x \in \bar{D}} u(x).$$

On the other hand, we find from (4.5)

$$e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \geqslant 2^{-1}.$$

Hence, also in this case,

$$u(0) \leqslant e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \max_{x\in \bar{D}} u(x),$$

and we obtain in all cases

$$u(0) \leqslant e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \max_{x\in \bar{D}} u(x).$$

The example constructed in §1.1, chap. I of paper [8] for $n = 2$ shows that it is impossible in this theorem to omit the restrictions, placed on the dimensions of the sphere Q_R. However, the theorem remains correct for arbitrary R, if, instead of equation (1.0.A), we consider equation (1.0.B). Actually, the condition $R < 1$ in theorem 4.1 is necessary only for the validity of lemma 3.1; it is not used elsewhere in the proof of this theorem. But lemma 3.2, analogous to lemma 3.1, is correct for equation (1.0.B) for any R. Thus, we have the theorem:

THEOREM 4.2. *Let the sphere Q_R of arbitrary radius R with centre at the point O contain a domain D, including the point O, and having limit points on the boundary of the sphere. Let Γ be that part of the boundary of the domain D, which is situated strictly inside Q_R. Let equation* (1.0.B) *be defined in D, and let*

$$\mu_n D = \sigma < \frac{\mu_n Q_R}{M},$$

where M is the constant of lemma 3.2.

Let, further, a positive solution of the equation be determined in D, continuous in $\bar{D}$, and vanishing on Γ.

Then

$$u(0) \leqslant e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \max_{x\in \bar{D}} u(x),$$

where C is a constant depending on the constant α of inequality (1.1), *the constant Δ of inequality* (3.8), *and the dimensionality n of space.*

§5. Theorem of Phragmén-Lindelöf type

We shall say that an unbounded domain D is of the "*type with solid angle size not larger than* η", if, for all integers m, beginning with a certain one, we have the inequality

$$\frac{\mu_n (D \cap Q_{2^m})}{\mu_n Q_{2^m}} < \eta, \tag{5.1}$$

where Q_{2^m} is the sphere of radius 2^m, with centre at the origin of coordinates Q.

THEOREM 5.1. *Let D be an unbounded domain, of the " type with solid angle size not larger than η ", in which equation (1.0.B) is defined. Let, further,*

$$\eta < \frac{2}{2^n M}, \tag{5.2}$$

where M is the constant of lemma 3.2.

Let there be determined in D a solution $u(x)$ of the equation, continuous in $\overline{D}$, and non-positive on the boundary of the domain D. Then, either 1) $u(x) \leqslant 0$ everywhere in D, or 2), if we put

$$M(R) = \sup_{|x|=R} u(x),$$

then

$$\liminf_{R\to\infty} \frac{M(R)}{\frac{1}{K\eta^{\frac{1}{n-1}}}} > 0,$$

where K is a constant depending on the constant α of the inequality (1.1), on the constant Δ of inequality (3.8), and on the dimensionality n of space.

PROOF. We suppose that there exists a point $\tilde{x} \in D$, such that $u(x) > 0$. We denote by G the component of the set of points $x \in D$, at which $u(x) > 0$, containing the point $\tilde{x}$.

Let inequality (5.1) be satisfied for the domain D for all m, beginning with m_0.

We put

$$M_m = \max_{\substack{|x|=2^{m-1}\\ x\in G}} u(x)$$

for all integers $m > m_1 = \max(m_0, [\log_2 |\tilde{x}|] + 1)$.

Let these maxima be attained respectively at the points

$$x^{(m)} \quad (m = m_1 + 1,\ m_1 + 2,\ \ldots).$$

For each integer $m > m_1$, we denote by $Q^{(m)}$ the sphere of radius 2^{m-1} with centre at the point $x^{(m)}$. We denote by G_m the component of the intersection $G \cap Q^{(m)}$ containing the point $x^{(m)}$.

From inequality (5.1), it follows that

$$\mu_n G_m < \omega_n 2^{mn} \eta,$$

where ω_n is the volume of the unit n-dimensional sphere, and from inequality (5.2), that

$$\mu_n G_m < \frac{\mu_n Q^{(m)}}{M}.$$

Applying theorem 4.2 to G_m and $Q^{(m)}$, we find

$$M_m < e^{-\frac{2^{(m-1)\frac{1}{n-1}}}{C\omega_n^{\frac{1}{n-1}} \cdot 2^{\frac{1}{n-1}} \eta^{\frac{1}{n-1}}}} \max_{x \in \bar{G}} u(x) = e^{-\frac{1}{2^{\frac{1}{n-1}} C\omega_n^{\frac{1}{n-1}} \eta^{\frac{1}{n-1}}}} \max_{x \in \bar{G}_m} u(x) \qquad (m = m_1 + 1,\ m_1 + 2,\ \dots).$$

Whence, according to the maximum principle, we get

$$M_m < e^{-\frac{1}{2^{\frac{n}{n-1}} C\omega_n^{\frac{1}{n-1}} \eta^{\frac{1}{n-1}}}} M_{m+1} < e^{-\frac{1}{4C\eta^{\frac{1}{n-1}}}} M_{m+1} \qquad (m = m_1 + 1,\ m_1 + 2,\ \dots). \tag{5.3}$$

Hence, in turn, we find that

$$M_m > e^{\frac{m - m_1}{4C\eta^{\frac{1}{n-1}}}} M_{m_1} \qquad (m = m_1 + 1,\ m_1 + 2,\ \dots).$$

By the maximum principle

$$M(R) > e^{\frac{m - m_1}{4C\eta^{\frac{1}{n-1}}}} M_{m_1} = (2^m)^{\frac{\log_2 e}{4C\eta^{\frac{1}{n-1}}}} e^{-\frac{m_1}{4C\eta^{\frac{1}{n-1}}}} M_{m_1}$$

where $2^{m-1} \leqslant R < 2^m$.

Hence

$$M(R) > a_0 R^{\frac{\log_2 e}{4C\eta^{\frac{1}{n-1}}}},$$

where $a_0 = M_{m_1} e^{-\frac{m_1}{4C\eta^{\frac{1}{n-1}}}}$ and $R > 2^{m_1}$.

Putting

$$K = \frac{4C}{\log_2 e},$$

we finally get

$$M(R)/R^{\frac{1}{K\eta^{\frac{1}{n-1}}}} > a_0 > 0 \text{ where } R > 2^{m_1},$$

which it was required to prove.

COROLLARY. *Let B be a n-dimensional solid angle at the origin of coordinates, cutting off, on the unit sphere, a domain, whose (n − 1)-dimensional area is equal to* η. *Let a solution u(x) of equation* (1.0.B) *be determined in B, and be non-positive on the boundary of B. There exist two constants* K_1 *and* K_2, *depending on the constant* α *of inequality* (1.1), *the constant* Δ *of inequality* (3.8), *and the dimensionality n of space, such that, if*

$$\eta < \frac{1}{K_1}$$

either $u(x) \leqslant 0$ *everywhere in B, or*

$$M(R) > R^{\frac{1}{K_2\eta^{\frac{1}{n-1}}}}$$

for sufficiently large R.

§6. Growth and decay of the solution at infinity and in the neighbourhood of a boundary point

THEOREM 6.1. *Let G be a domain, situated outside some sphere Q with centre at the origin of coordinates. Let G be of the "type with solid angle size not larger than* η*". Let, further,* Γ *be that part of the boundary of the domain situated strictly outside the sphere Q. Let equation (1.0.B) be defined in G, and let* η *satisfy the inequality*

$$\eta < \frac{1}{2^n M} \tag{6.1}$$

(M is the constant of lemma 3.1).

Let there be determined in G a solution u(x) of the equation, positive inside the domain, and vanishing on Γ*. Then, if one puts*

$$M(R) = \sup_{\substack{|x|=R \\ x \in G}} u(x),$$

either

$$\liminf_{R\to\infty} \left(M(R)/R^{\frac{1}{K\eta^{\frac{1}{n-1}}}}\right) > 0,$$

or

$$\limsup_{R\to\infty} \left(M(R)\cdot R^{\frac{1}{K\eta^{\frac{1}{n-1}}}}\right) < \infty,$$

where K is a constant, depending on α *in inequality (1.1),* Δ *of inequality (3.8), and on n.*

The proof is similar to the proof of the theorem of the previous section. Let $x^{(m)}$, $Q^{(m)}$, G_m ($m = m_1 + 1$, $m_1 + 2$, ...) have the same meanings as in the previous section. Then, as there, we find that for sufficiently large m ($m > m_1$)

$$M_m < e^{-\frac{1}{2^{\frac{n}{n-1}} C \omega_n^{\frac{1}{n-1}} \eta^{\frac{1}{n-1}}}} \max_{x\in \bar{G}} u(x) \qquad (m = m_1 + 1,\ m_1 + 2,\ \ldots). \tag{6.2}$$

Unlike what was done in §5, we cannot hence conclude that the inequality (5.3) is valid.

Here we proceed as follows: from the maximum principle and (6.2) there follows the validity of at least one of the inequalities

$$M_m < e^{-\frac{1}{4C\eta^{\frac{1}{n-1}}}} M_{m+1} \tag{6.3}$$

or

$$M_m < e^{-\frac{1}{4C\eta^{\frac{1}{n-1}}}} M_{m-1}. \tag{6.4}$$

Here, if for some m the inequality (6.3) is satisfied, then for all larger m this inequality will also be true. Thus, either for all $m = m_1 + 1$, $m_1 + 2$, ... the inequality (6.4) is valid, or for all m, larger than some m_2, the inequality (6.3) is correct.

Hence, either

$$M_m > e^{\frac{m-m_2}{4C\eta^{\frac{1}{n-1}}}} M_{m_2} \qquad (m = m_2 + 1,\ m_2 + 2,\ \ldots),$$

or

$$M_m < e^{\frac{m_1-m}{4C\eta^{\frac{1}{n-1}}}} M_{m_1} \qquad (m = m_1 + 1,\ m_1 + 2,\ \ldots).$$

In the first case

$$M(R)/R^{\frac{1}{K\eta^{\frac{1}{n-1}}}} > a_2 \qquad \text{for } R > 2^{m_2},$$

in the second

$$M(R)\, R^{\frac{1}{K\eta^{\frac{1}{n-1}}}} < a_1 \quad \text{for} \quad R > 2^{m_1},$$

where

$$K = \frac{4C}{\log_2 e}, \qquad a_1 = M_{m_1} e^{\frac{m_1}{4C\eta^{\frac{1}{n-1}}}}, \qquad a_2 = M_{m_2} e^{\frac{m_2}{4C\eta^{\frac{1}{n-1}}}},$$

which it was required to prove.

One can obtain a theorem analogous to theorem 6.1, characterizing the behaviour of the solution in the neighbourhood of a finite boundary point of the domain, in dependence on the size of the part of the domain lying in the sphere with centre at this boundary point, when the radius of the sphere tends to zero.

THEOREM 6.2. *Let G be a domain, having the point O as a limit point. Let equation (1.0.A) be defined in G.*

Further, let a number η exist, satisfying the inequality

$$\eta < \frac{1}{2^n M},$$

where M is the constant of lemma 3.1, and such that, if, for the integer m, $Q_{2^{-m}}$ denotes the sphere of radius 2^{-m} with centre at O, then

$$\frac{\mu_n (G \cap Q_{2^{-m}})}{\mu_n Q_{2^{-m}}} < \eta$$

for all m, beginning with a certain one.

Let a solution of the equation, determined in G, be positive inside the domain and vanish on the part of the boundary situated in the neighbourhood of the point O, except at the point O itself. Then, if we put

$$M(r) = \sup_{\substack{|x|=r \\ x \in G}} u(x),$$

either

$$\liminf_{r \to 0} \left(M(r)\, r^{K\eta^{\frac{1}{n-1}}}\right)^{\frac{1}{\frac{1}{}}} > 0,$$

or

$$\limsup_{r \to 0} \left(M(r)/ r^{K\eta^{\frac{1}{n-1}}}\right)^{\frac{1}{\frac{1}{}}} < \infty,$$

where K is a constant depending on the constant α of inequality (1.1), the constant Δ of inequality (3.1), and on the dimensionality n of space.

The proof is almost a word for word repetition of the proof of theorem 6.1, but, since it is possible to employ lemma 3.1 here, instead of lemma 3.2, the statement proves to be true for equation (1.0.A).

§7. The principal lemma for the self-adjoint equation

We shall consider the equation

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) = 0, \tag{1.0.C}$$

defined in some domain D. We shall suppose nothing about the coefficients of this equation, beyond that they satisfy inequality (1.1), are measurable, and have their moduli bounded by unity.

We describe as a solution of this equation, a function $u(x) \in W_2^1$ continuous in D, and satisfying the integral identity

$$\int_{D'} \sum_{i,k=1}^{n} a_{ik}\frac{\partial u}{\partial x_k}\frac{\partial \varphi}{\partial x_i}\,dx = 0, \tag{7.1}$$

where D' is an arbitrary domain with smooth boundary, contained together with its boundary in D, and $\varphi(x)$ is an arbitrary function of W_2^1 continuous in D', vanishing on the boundary of D'.

LEMMA 7.1. *Let there be situated in the sphere Q_R of arbitrary radius R a domain D, containing O, the centre of the sphere and having limit points on the boundary of the sphere S_R. Let Γ be that part of the boundary of D, which lies strictly inside the sphere Q_R.*

There exists a constant M, depending only on the constant α of inequality (1.1), and on the dimensionality n of space, such that, if

$$\mu_n D < \frac{\mu_n Q_R}{M} \tag{7.2}$$

and equation (1.0.C) *is defined in* D, *then, for any positive solution* $u(x)$ *of it, determined in* D, *continuous in* $\overline{D}$, *and vanishing on* Γ, *we have the inequality*

$$u(0) < \frac{1}{2} \max_{x \in \overline{D}} u(x). \tag{7.3}$$

1°. We prove first of all, that, for the validity of this lemma, it is sufficient to prove it for equation (1.0.C), the coefficients of which a_{ik} are n times continuously differentiable.

We shall denote such an equation by (1.0.D).

In fact, let the lemma be proved for equation (1.0.D), and let there be given the equation (1.0.C).

Let equation (1.0.C) satisfy inequality (1.1) for some constant α. Let the domain D be such that it satisfies the inequality (7.2) with the constant $2M$, where M is the constant in inequality (7.2) necessary for the validity of the lemma for equation (1.0.D) for $\alpha/2$.

Let $u(x)$ be a solution of equation (1.0.C), which is positive in D, continuous in $\overline{D}$, and vanishing on Γ.

We assign an arbitrary $\varepsilon > 0$, and take domains D' and D'', $D' \subset D'' \subset \overline{D}'' \subset D$, with twice smooth boundaries, sufficiently near to D, so that

1) $O \in D'$;

2) the value of u at each point of the boundary of D' differs from the value of u at the nearest point of the boundary of D by less than ε;

3) if R' denotes the upper bound of the distance to the point O from points belonging to D', then

$$\mu_n D' < \frac{\mu_n Q_{R'}}{M}, \tag{7.4}$$

where $Q_{R'}$ is the sphere of radius R', with centre at the point O.

We construct a sequence of equations, defined in $\overline{D}''$:

$$L^{(m)} u^{(m)} \equiv \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ik}^{(m)} \frac{\partial u^{(m)}}{\partial x_k} \right) = 0 \qquad (m = 1, 2, \ldots),$$

such that their coefficients are n-times continuously differentiable, and satisfy the conditions

$$\sum_{i,k=1}^{n} a_{ik}^{(m)} \xi_i \xi_k \geqslant \frac{\alpha}{2} \sum_{i=1}^{n} \xi_i^2, \qquad |a_{ik}^{(m)}| < 1$$

and

$$\int_{D'} (a_{ik} - a_{ik}^{(m)})^2 \, \partial x \to 0 \quad \text{as} \quad m \to \infty. \tag{7.5}$$

This sequence can be constructed, for example, by means of averaging the coefficients of the original equation.

We denote by γ the boundary of the domain D''.

Let $u^{(m)}(x)$ be a solution of the equation

$$L^{(m)}u^{(m)}=0$$

in D'', satisfying the boundary conditions

$$u^{(m)}|_\gamma = u|_\gamma .$$

The family $\{u^{(m)}\}$, by the maximum principle, is uniformly bounded, and by a theorem of Giorgi [3], is equicontinuous in $\overline{D}'$; hence there exists a sequence $u^{(m_k)}(x)$ ($k = 1, 2, \ldots,$), converging in $\overline{D}'$ to some function u^*. We prove that $u^* \equiv u$. It is sufficient for this to prove that the sequence $u^{(m)}$ converges to u in the mean.

This is, however, actually so. In fact, we have

$$\int_{D''} \sum_{i,k=1}^{n} a_{ik}^{(m)} \frac{\partial u^{(m)}}{\partial x_k} \frac{\partial}{\partial x_i}(u-u^{(m)})\,dx = 0$$

and

$$\int_{D''} \sum_{i,k=1}^{n} a_{ik} \frac{\partial u}{\partial x_k} \frac{\partial}{\partial x_i}(u-u^{(m)})\,dx = 0.$$

Whence

$$\int_{D''} \sum_{i,k=1}^{n} a_{ik} \frac{\partial (u-u^{(m)})}{\partial x_i} \frac{\partial (u-u^{(m)})}{\partial x_k}\,dx = \int_{D''} \sum_{i,k=1}^{n} (a_{ik}-a_{ik}^{(m)}) \frac{\partial u^{(m)}}{\partial x_k} \frac{\partial}{\partial x_i}(u-u^{(m)})\,dx, \tag{7.6}$$

and since there exists a constant C, such that

$$\left\| \frac{\partial u^{(m)}}{\partial x_k} \right\|_{L^2_{D''}} < C \qquad (m = 1, 2, \ldots; k = 1, \ldots, n),$$

and besides, from the condition $\left\| \frac{\partial u}{\partial x_k} \right\|_{L^2_{D''}} < \infty$, on applying to the right-hand side of (7.6) Schwarz's inequality, and using (7.5), we find that

$$\left\| \frac{\partial (u-u^{(m)})}{\partial x_k} \right\|_{L^2_{D''}} \to 0 \quad \text{as} \quad m \to \infty,$$

whence it follows that

$$\| u-u^{(m)} \|_{L^2_{D'}} \to 0 \quad \text{as} \quad m \to \infty.$$

Let ε be sufficiently small so that $u(0) > \varepsilon$. Then, for sufficiently large m

$$u^{(m)}(0) > \varepsilon.$$

For each such m we take the set of points $x \in D'$ at which $u^{(m)}(x) > \varepsilon$. We denote by $D^{(m)}$ the component of this set containing the point O. From the inequality (7.4), it follows that

$$\mu_n D^{(m)} < \frac{\mu_n Q_{R'}}{M},$$

and then, for the function

$$\widetilde{u}^{(m)} = u^{(m)} - \varepsilon$$

by our hypothesis that the lemma holds for equation (1.0.D), it follows that

$$\max_{x\in\overline{D}'} \widetilde{u}^{(m)}(x) \geqslant 2\widetilde{u}^{(m)}(0),$$

i.e.

$$\max_{x\in\overline{D}} u \geqslant \max_{x\in D'} u(x) \geqslant 2u(0) - \varepsilon,$$

and, because ε was arbitrary,

$$\max_{x\in\overline{D}} u(x) \geqslant 2u(0),$$

and our statement is proved.

2^{0}. Thus, we shall prove the lemma for equation (1.0.D). In this paragraph we carry out a further reduction: we denote by D_0 the set of points $x \in D \cap Q_{\frac{R}{2}}$, where $Q_{\frac{R}{2}}$ is the sphere of radius $\frac{1}{2}R$ with centre at the point O, at which $u > \frac{1}{2}u(0)$. Besides the inequality (7.2), let there be satisfied the further inequality

$$\mu_n D_0 \geqslant \frac{\mu_n Q_R}{4^n R}, \tag{7.7}$$

and let the lemma be true under this hypothesis. We prove that it is true without the hypothesis (7.7), if we take a larger constant: $2^n M$ instead of M.

Suppose that for the domain D condition is not satisfied. For each $r (0 < r < R)$, we put

$$M(r) = \max_{\substack{|x|=r \\ x\in D}} u(x) = u(x_r)$$

(considering the origin of coordinates to be the point O).

We prove that for any r $\left(0 < r \leqslant \frac{R}{2}\right)$, there exists Δ $\left(0 < \Delta \leqslant \frac{R}{2}\right)$, such that

$$M(r+\Delta) \geqslant M(r)\left(1 + \frac{2\Delta}{R}\right). \tag{7.8}$$

We introduce the following notation:

Q_r^m is the sphere of radius $R/2^{m+1}$ with centre at the point x_r;

B_r^m is the set of points $x \in D$, where $u(x) > \frac{2^m - 1}{2^m} M(r)$;

D_r^m is the component of the intersection $B_r^m \cap Q_r^m$, which contains the point x_r;

$$u_r^m = u(x) - \frac{2^m - 1}{2^m} M(r).$$

Since, by our hypothesis, condition (7.2) is satisfied for the domain D with the constant $2^n M$, then

$$\mu_n D_r^1 < \frac{\mu_n Q_r^1}{M}$$

Suppose, further, that for some m, condition (7.7) is not satisfied for the domain D_r^m and the sphere Q_r^m. Then

$$\mu_n D_r^{m+1} < \frac{\mu_n Q_r^{m+1}}{M} .$$

Finally, there exists m_0 such that for the domain $D_r^{m_0}$ and the sphere $Q_r^{m_0}$ the inequality (7.7) is satisfied. This can be proved as follows. At the point x_r we have grad $u \neq 0$. Actually, x_r is on the surface of the sphere Q_r of radius r with centre at the point O, and at it, according to the maximum principle, there is taken the largest value of those, which are taken in that part of the domain which lies inside Q_r, according to the strict form of the maximum principle $u(x_r)$ being strictly larger than the values taken by u inside $Q_r \cdot D$. But from this it follows that (cf. [10]) at the point x_r, $\frac{\partial u}{\partial r} > 0$.

The function u, being a solution of equation (1.0.D), is twice continuously differentiable (and even a greater number of times, but this is immaterial at present). Hence the level surface, passing through the point x_r, is in its neighbourhood a twice continuously differentiable surface. Therefore there exists m_0, such that there is a sphere of radius $R/2^{m_0+r}$, with the point x_r on its surface, and which at all remaining points lies in the domain given by $u(x) > M(r)$. But then this sphere is in D_r^m for any m, and hence

$$\mu D_r^{m_0} > \frac{\mu Q_r^{m_0}}{2^n} ,$$

i.e. for this m_0 inequality (7.7) is satisfied.

From all this it follows that there exists m_1 such that

$$\mu_n D_r^{m_1} < \frac{\mu_n Q_r^{m_1}}{M} ,$$

and the inequality (7.7) is satisfied for the domain $D_r^{m_1}$ and the sphere $Q_r^{m_1}$.

By our hypothesis, the lemma is true in this case. We apply it to the function $u_r^{m_1}$, and obtain

$$\max_{x \in \bar{D}_r^{m_1}} u_r^{m_1}(x) \geqslant 2u_r^{m_1}(x_r) = \frac{M(r)}{2^{m_1-1}} ,$$

or

$$M\left(r + \frac{R}{2^{m_1+1}}\right) \geqslant \max_{x \in D_r^{m_1}} u_r^{m_1}(x) + \frac{2^{m_1}-1}{2^{m_1}} M(r) \geqslant M(r)\left(1 + \frac{1}{2^{m_1}}\right)$$

It remains to put $\Delta = R/2^{m_1+1}$.

But, from inequality (7.8), it follows that

$$\max_{x \in \bar{D}} u(x) \geqslant 2u(0).$$

Actually, let r_1 be the upper bound of all $r \leqslant R$, such that

$$M(r) \geqslant u(0)\left(1+\frac{2r}{R}\right). \tag{7.9}$$

If $r_1 < \frac{1}{2}R$, then

$$M(r_1+\Delta) \geqslant M(r_1)\left(1+\frac{2\Delta}{R}\right) > u(0)\left(1+\frac{2(r_1+\Delta)}{R}\right),$$

and we arrive at a contradiction to the hypothesis that r_1 is the upper bound of r, for which (7.9) is satisfied. Thus, $r_1 \geqslant \frac{1}{2}R$, and this means that

$$M(r_1) \geqslant 2u(0),$$

and our assertion is proved.

3°. We pass on to the proof proper of lemma 7.1. In accordance with 1° and 2° we shall prove it for equation (1.0.D), under the additional condition (7.7).

Since equation (1.0.D) is homogeneous, and the magnitude of the derivatives of the coefficients does not interest us, we can make a similarity transformation, and instead of the sphere Q_R of arbitrary radius consider a sphere of any fixed radius. It is convenient for us to consider the sphere of radius 2. We denote it by Q_2, and denote the concentric sphere of radius 1 by Q_1.

Let inequality (7.2) be satisfied for some M. Let a solution of equation (1.0.D) be determined in D, which is positive in D, continuous in $\overline{D}$, and which vanishes on Γ, and let inequality (7.7) be satisfied. We show that for sufficiently large M (depending on α and n) (7.3) is correct.

We now make use of the property of $u(x)$, as a solution of equation (1.0.D), being $n + 2$ times differentiable (it is essential for us that it be n times differentiable). By a theorem of the paper of Kronrod and Landis [11], an n-times differentiable function of n variables has the property that the set of points of the domain of definition, where the gradient of this function vanishes, is mapped on the number-axis in a set of measure zero.

We put $D_1 = D \cap Q_1$. Further, for every t $(0 < t < u(0))$, we denote by B_t the set of points $x \in D$, at which $u(x) > t$, and put $G_t = B_t \cap Q_1$. We denote by γ_t, that part of the boundary of the set G_t, which lies strictly inside Q_1. Consider the integral

$$I(t) = \int_{\gamma_t} \frac{d\sigma}{\frac{\partial u}{\partial n}},$$

where n is the inward normal to G_t. By the quoted theorem of Kronrod-Landis, for nearly all t $(0 < t < u(0))$, γ_t is a smooth $(n-1)$-dimensional manifold $\frac{\partial u}{\partial n} > 0$, and thus for nearly all t, this integral has a meaning, and is positive (or equals ∞).

Let us now consider $\displaystyle\int_0^{\frac{u(0)}{2}} I(t)\,dt.$

This integral does not exceed the volume of the domain D_1, and hence, by inequality (7.2),

$$\int_0^{\frac{u(0)}{2}} I(t)\,dt \leqslant \mu_n D_1 < \mu_n D < \frac{\mu_n Q_1}{M} = \frac{\omega_n}{M}$$

(ω_n is here the volume of the n-dimensional unit sphere).

Whence it follows that there exists t_0 such that

$$I(t_0) < \frac{2\omega_n}{Mu(0)},$$

i.e.

$$\int_{\gamma_{t_0}} \frac{d\sigma}{\frac{\partial u}{\partial n}} < \frac{2\omega_n}{Mu(0)}.$$

One can here select t_0, so that the set of the level $u(x) = t_0$ does not contain points at which grad $u(x) = 0$.

Applying Schwarz's inequality, we find then

$$\int_{\gamma_{t_0}} \frac{\partial u}{\partial n}\,d\sigma > \frac{Mu(0)}{2\omega_n}(\mu_{n-1}\gamma_{t_0})^2. \tag{7.10}$$

We now use the property that, for a domain lying inside an n-dimensional sphere with volume less than the volume of the sphere divided by 2^n, it is correct that the part of its boundary lying strictly inside the sphere has $(n-1)$-dimensional measure not less than $1/2^n$ times the entire measure of the boundary of the domain.

If we suppose that

$$M > 2^n,$$

and denote by Γ_{t_0} the whole boundary of G_{t_0}, then, by what has been said

$$\mu_{n-1}\gamma_{t_0} > \frac{1}{2^n}\,\mu_{n-1}\Gamma_{t_0},$$

and, by the isoperimetric inequality,

$$\mu_{n-1}\gamma_{t_0} > \frac{1}{2^n}(\mu_n G_{t_0})^{\frac{n-1}{n}}.$$

Since, by inequality (7.7)

$$\mu_n G_{t_0} > \frac{\omega_n}{4^n M},$$

then

$$\mu_{n-1}\gamma_{t_0} > \frac{\omega_n^{\frac{n-1}{n}}}{2^{3n-2} M^{\frac{n-1}{n}}},$$

and we get from (7.10)

$$\int_{\gamma_{t_0}} \frac{\partial u}{\partial n}\,d\sigma > \frac{u(0)\omega_n^{\frac{n-2}{n}}}{2^{6n-5} M^{\frac{n-2}{n}}}. \tag{7.11}$$

We denote by δ_{t_0} that part of the boundary of B_{t_0}, which lies strictly inside Q_2. Then

$$\gamma_{t_0} \subset \delta_{t_0}$$

We denote by C_{t_0} the part of B_{t_0} situated in the spherical layer $Q_2 \setminus Q_1$. Let Σ be some piecewise smooth surface, separating in C_{t_0} the sphere $|x| = 1$ from the sphere $|x| = 2$ (any continuous curve having limit points on both spheres certainly intersects Σ).

Thus, Σ together with part of δ_{t_0} bounds some subdomain of the domain B_{t_0}, containing G_{t_0}. We denote it by G. The boundary γ of this domain includes in itself all γ_{t_0}, and besides, contains points belonging to δ_{t_0} and Σ. Let $\gamma' = (\gamma \cap \delta_{t_0}) \setminus \gamma_{t_0}$, and $\gamma'' = \gamma \cap \Sigma$.

We consider the equation

$$\int_G \sum \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) dx = 0$$

and applying Green's formula to the left-hand side, we obtain

$$\int_\gamma \frac{\partial u}{\partial \nu} d\sigma = \int_{\gamma_{t_0}} \frac{\partial u}{\partial \nu} d\sigma + \int_{\gamma'} \frac{\partial u}{\partial \nu} d\sigma + \int_{\gamma''} \frac{\partial u}{\partial \nu} d\sigma = 0,$$

where $\frac{\partial}{\partial \nu}$ is the derivative along the normal ($\frac{\partial}{\partial \nu} = \sum_{i,k=1}^{n} a_{ik}\gamma_i \frac{\partial}{\partial x_k}$, γ_i being the direction cosines of the normal).

Since

$$\int_{\gamma_{t_0}} \frac{\partial u}{\partial \nu} d\sigma \geqslant \alpha \int_{\gamma_{t_0}} \frac{\partial u}{\partial n} d\sigma$$

and $\left.\frac{\partial u}{\partial \nu}\right|_{\gamma'} > 0$, then

$$\alpha \int_{\gamma_{t_0}} \frac{\partial u}{\partial n} d\sigma < \left| \int_{\gamma''} \frac{\partial u}{\partial \nu} d\sigma \right| \leqslant \int_\Sigma \left| \frac{\partial u}{\partial \nu} \right| d\sigma$$

and, applying inequality (7.11), we find

$$\int_\Sigma \left| \frac{\partial u}{\partial \nu} \right| d\sigma > \frac{\alpha u(0)\, \omega_n^{\frac{n-2}{n}}}{2^{6n-5} M^{\frac{n-2}{n}}}. \tag{7.12}$$

Our lemma will be proved, if we are able to show that the surface Σ can be always selected so that

$$\int_\Sigma \left| \frac{\partial u}{\partial \nu} \right| d\sigma < \frac{C}{M} \operatorname*{osc}_{x \in D} u(x),$$

where C is a constant depending on α and on the dimensionality of space. This is a consequence of a general theorem of analysis (in a certain sense analogous to the theorem of Lagrange about the derivative at a point of an interval for a function of one variable).

This theorem will be proved in the next section, and thus the proof of lemma 7.1 will be completed.

§8. A theorem of analysis

The theorem of this section was proved by M.L. Gerver and myself [12].

THEOREM 8.1. *Let a domain D, $\mu_n D < 2^n \omega_n/M$, be situated in the spherical layer $S = \{1 < |x| < 2\}$. Let the domain D have limit points on both spheres $|x| = 1$ and $|x| = 2$, and let that part of its boundary, which is situated strictly inside D, be a smooth surface.*

Let there be defined the quadratic form $\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k$ in the domain D, satisfying the inequalities

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \geqslant \alpha \sum_{i=1}^{n} \xi_i^2, \quad \alpha > 0, \quad |a_{ik}| < 1, \tag{8.1}$$

the coefficients of this quadratic form being continuously differentiable functions in D.

Further, let there be defined in $\overline{D}$ a function $f(x)$, twice continuously differentiable, and satisfying the condition

$$\operatorname*{osc}_{x \in \overline{D}} f(x) < 1. \tag{8.2}$$

Then, there exists a piecewise smooth surface Σ, separating in D the sphere $|x| = 1$ from $|x| = 2$, such that

$$\int_{\Sigma} \left| \sum_{i,k=1}^{n} a_{ik}\gamma_i \frac{\partial f}{\partial x_k} \right| d\sigma < \frac{C}{M}, \tag{8.3}$$

where C is a constant depending on the constant α of inequality (8.1) and on the dimensionality n of space (γ_i being the direction cosines of the normal to the surface).

For the proof of the theorem we require the lemma:

LEMMA 8.1. *We denote by Ω the set of points $x \in \overline{D}$, where* grad $f = 0$. *Then, this set can be included in a finite number of spheres $Q_1, \ldots, Q_N$, such that, if S_m denotes the surface of the m-th sphere*

$$\sum_{m=1}^{N} \int_{S_m} |\operatorname{grad} f|\, d\sigma < 1.$$

PROOF. We divide the set Ω into two parts: we attribute to the set Ω' those points of Ω, in which the second differential $d^2 f$ is not zero, and to the set Ω'' those points of Ω where $d^2 f = 0$. The set Ω' has measure zero, since all points at which it is dense belong to Ω''.

We take some $\varepsilon > 0$, and include Ω' in an open set G, of measure less than ε.

We cover each point $x \in \Omega'$ by a sphere K'_x with centre at this point and contained in G, and let S'_x be the surface of the sphere Q'_x concentric with K'_x with diameter five times as large. We evaluate

$$\int_{S'_x} |\operatorname{grad} f|\, d\sigma.$$

Since f is twice continuously differentiable in $\overline{D}$, then all its second derivatives have bounded moduli. Let L be a constant bounding them. Since at the point x we have grad $f = 0$, then $\|\text{grad } f\|_{S'_x} < Lr_x$, where r_x is the radius of S'_x. Whence

$$\int\limits_{S'_x} |\operatorname{grad} f|\, d\sigma < Lr_x\mu_{n-1} S'_x = 5^n nL\mu_n K'_x. \tag{8.4}$$

We select from the aggregate of spheres $\{K'_x\}$ a countable set $K'_1, \ldots, K'_m, \ldots$, such that these spheres are mutually disjoint, and the (open) spheres concentric with them with five times the radius

$$Q_1, \ldots, Q'_m, \ldots$$

cover all the set Ω'. This can be done by applying the process of Banach: we take from $\{K'_x\}$ a sphere whose diameter exceeds half the upper bound of the diameters. We denote it K'_1. We reject all the spheres intersecting it. From the remaining spheres we take a sphere whose diameter is greater than half the upper bound of the remaining spheres. We denote it by K'_2, etc.

By reason of (8.4) we get

$$\sum_{m=1}^{\infty} \int\limits_{S'_m} |\operatorname{grad} f|\, d\sigma < 5^n nL\varepsilon. \tag{8.5}$$

We denote here by S'_m the surface of the sphere Q'_m.

Let now $x \in \Omega''$. Since at this point both first and second differentials vanish, then there exists a sphere Q''_x with centre at the point x, such that everywhere on its surface

$$|\operatorname{grad} f| < \varepsilon r_x,$$

where r_x is the radius of this sphere. In this connection one can select the sphere so that $r_x < 1$. Denoting by K''_x the sphere concentric with, and radius one fifth that of, Q''_x. Then

$$\int\limits_{S''_x} |\operatorname{grad} f|\, d\sigma < \varepsilon r_x\mu_n S''_x = 5^n n\varepsilon\mu_n K''_x. \tag{8.6}$$

Just as before, we select from the aggregate $\{K''_x\}$ of spheres a countable number

$$K''_1, \ldots, K''_m, \ldots$$

of mutually disjoint spheres such that the concentric spheres with five times the radius

$$Q''_1, \ldots, Q''_m, \ldots$$

cover all Ω''.

In view of (8.6), and considering that K''_m does not go outside the limit of the sphere $|x| \leqslant 2 + \frac{1}{5}$, we get

$$\sum_{m=1}^{\infty} \int\limits_{S''_m} |\operatorname{grad} f|\, d\sigma < 5^n n\varepsilon\omega_n \left(2 + \frac{1}{5}\right)^n = 11^n n\omega_n\varepsilon; \tag{8.7}$$

S''_m is here the surface of the sphere Q''_m, ω_n is the volume of the unit

n-dimensional sphere. Combining the spheres $\{Q'_m\}$ and $\{Q''_m\}$ we get an aggregate of open spheres, covering the closed set $\Omega = \Omega_1 \bigcup \Omega_2$. We select from them a finite number. Let these spheres be $Q_1, \ldots, Q_N$, and their surfaces be $S_1, \ldots, S_N$ respectively.

If we select $\varepsilon = 1/(5^n nL + 11^n n\omega_n)$, then, from inequalities (8.5) and (8.7), we get

$$\sum_{m=1}^{N} \int_{S_m} |\operatorname{grad} f| \, d\sigma < 1,$$

which it was required to prove.

We now proceed to the proof of the theorem.

First of all we find, corresponding to the lemma, spheres $Q_1, \ldots, Q_N$, and exclude them from the domain D. We put

$$D^* = \overline{D} \setminus \sum_{m=1}^{n} Q_m.$$

We form the intersection of D^* with the closed spherical layer $1\frac{1}{8} \leqslant |x| \leqslant 1\frac{7}{8}$. We denote this intersection by D'.

D' is a closed set, and everywhere in D', grad $f = 0$. Hence, $|\operatorname{grad} f| > \beta$, $\beta > 0$.

We continue $f(x)$ from the set D' to some neighbourhood of it in an arbitrary twice continuously differentiable manner.

We continue also all coefficients a_{ik} of the quadratic form to some neighbourhood of the set D' in an arbitrary continuously diffentiable manner. We take now the δ-neighbourhood D'_δ of the set D' with δ ($< 1/8$) sufficiently small so that in $\overline{D}'_\delta$ the continuation of f is defined (we denote it by the same letter f); f satisfies in D'_δ the condition (8.2), and $|\operatorname{grad} f| > \beta$ in D'_δ, besides, δ is sufficiently small so that in $\overline{D}'_\delta$ the continuation of the coefficients of the quadratic form is determined (we denote them also by a_{ik}), and satisfy inequality (8.1) there, and finally, $\mu_n D'_\delta < 2^n \omega^k/M$.

In D'_δ we consider the system of ordinary differential equations:

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik} \frac{\partial f}{\partial x_n} \qquad (i+1, \ldots, n). \tag{8.8}$$

We consider the scalar product of the vector of the right-hand side with grad f. We have

$$\left(\sum_{h=1}^{n} a_{ik} \frac{\partial f}{\partial x_k}, \ \operatorname{grad} f\right) = \sum_{i,k=1}^{n} a_{ik} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_k} \geqslant \alpha \, |\operatorname{grad} f|^2.$$

From this inequality follows, firstly, that in the domain D'_δ there are no stationary points of the system (8.8) (since grad $f \neq 0$ in it), and secondly, that the direction of the field forms with the direction of the gradient at the given point an angle not a right angle, the modulus of the cosine of this angle being larger than a constant, depending on α: Let $I(x)$ be the direction of the field at the point x; then

$$|\cos(\widehat{\mathbf{l}(x), \operatorname{grad} f})| > C(\alpha) > 0. \tag{8.9}$$

If $\frac{\partial f}{\partial l}$ denotes the derivative in the direction of the field, then, remembering that $|\operatorname{grad} f| > \beta$ in D'_δ, we get from (8.9)

$$\left|\frac{\partial f}{\partial l}\right| > C(\alpha)\,|\operatorname{grad} f| > C(\alpha)\,\beta. \tag{8.10}$$

It hence follows that in D'_δ there are no closed trajectories of the system (8.8), and all trajectories have uniformly length, and are described in uniformly bounded time. Let L be a constant, bounding the length of the trajectories, and T a constant, bounding the time of motion on a trajectory.

Let the surface S be tangent to the direction of the field at each point of it. Then

$$\int_S \left| \sum_{i,k=1}^{n} a_{ik}\gamma_i \frac{\partial f}{\partial x_k} \right| d\sigma = 0,$$

since the integrand is identically zero (here, as before, γ_i are the direction cosines of the normal to S).

We use these in the construction of the surface Σ, which we require. The ruled surfaces, whose generators are trajectories of the system (8.8), form the basis of Σ. In the integral of interest to us, they contribute nothing. These surfaces will have the form of walls of thin pipes, which overlap and cover all D'. Then, in some pipes we insert partitions. On these partitions our integral will not be equal to zero, but we shall be able to make it not very large. We now begin the construction of the pipes.

We shall find a number η $(0 < \eta < \frac{1}{2}\delta)$, satisfying the following conditions: for any pair of points $x_1, x_2 \in D'_\delta$, for which $|x_1 - x_2| < \eta$:

1) there is satisfied the inequality

$$|\operatorname{grad} f(x_1) - \operatorname{grad} f(x_2)| < \frac{\beta}{2}; \tag{8.11}$$

2) the angle between the directions of the field at x_1 and x_2 is less than $\frac{1}{8}\pi$;

2) let y_1 and y_2 be points, at which we find ourselves moving on the trajectories from the points x_1 and x_2 respectively at the same time $t < T$. Then $|y_1 - y_2| < \frac{1}{2}\delta$.

Let the number η, besides, be so small that the following condition holds:

4) let x_1 and x_2 be two arbitrary points on one trajectory of the system (8.8). Through each of these points draw an $(n - 1)$-dimensional hyperplane orthogonal to the trajectory. Then, if one takes in each of these hyperplanes a $(n - 1)$-dimensional sphere, with centre at x_1 and x_2 respectively, and radius equal to η, these spheres do not intersect.

Condition 1) can be satisfied because of the uniform continuity of grad grad f in D'_δ. Condition 2) can be satisfied by the uniform continuity of the field. The possibility of satisfying 3) follows from the uniformly continuous dependence of the solutions of the system (8.8) on the initial conditions. Finally, condition 4) can be satisfied since the right-hand sides of the system (8.8) are continuously differentiable in $\overline{D'_\delta}$, and hence the trajectories have uniformly bounded curvature.

We now find ζ_1 such that, for any pair of points x_1, $x_2 \in D'$ satisfying the inequality $|x_1 - x_2| < \zeta_1$, the trajectory passing through the point x_2 does not leave the $\frac{\eta}{2}$-neighbourhood of the trajectory passing through through the point x_1.

Further, we take a number ζ_2 $(0 < \zeta_2 < \zeta_1)$, such that for any point $x_1 \in D'$, and any point x_2 on the $(n - 1)$-dimensional hyperplane passing through the point x_1 and perpendicular to the trajectory through x_1, whenever the distance between the points x_1 and x_2 is not less than $\frac{1}{2}\zeta_1$, it follows that the trajectory passing through x_2 lies outside the ζ_2-neighbourhood of the trajectory passing through x_1.

The existence of numbers ζ_1 and ζ_2 with these properties follows from the uniform boundedness of the lengths of the trajectories (by the constant L).

We introduce the following definitions.

We consider some point $x \in D'$, and draw through it a trajectory l. We denote the $\frac{\delta}{2}$-neighbourhood of D' by $D'_{\frac{\delta}{2}}$.

The trajectory l at both ends reaches the boundary of D'_δ, and hence intersects the boundary of $D'_{\frac{\delta}{2}}$. We denote the points of intersection of l with the boundary of $D'_{\frac{\delta}{2}}$, which are nearest (on the curve l) to the point x, by x_1 and x_2: the point x_2 being situated from x on the side corresponding to a positive change in t, and x_1 on the negative side. Suppose that we reach the point x_2 from x in time t_2, and the point x_1 from the point x in time t_1. We draw an $(n - 1)$-dimensional hyperplane through x orthogonal to l, and in this hyperplane we take a domain ω with piecewise smooth boundary σ, lying inside the sphere of radius ζ_1 with centre at the point x. Through each point $y \in \omega$, we draw a trajectory, and denote by l_y the portion of this trajectory, which is covered on moving along the trajectory from the point y in the positive direction for a time t_2, and in the negative direction for a time t_1. The set-theoretic sum T of the curves l_y, when y covers ω, is called the *pipe generated by* ω $(T = \bigcup_{y \in \omega} l_y)$.

In the same way, we define l_y for $y \in \sigma$. The set-theoretic sum C of the curves l_y, when y describes σ $(C = \bigcup_{y \in \sigma} l_y)$ is called the *wall* of the pipe T. The piece of the trajectory l between x_1 and x_2 is called the *axis* of the pipe T. We note that the axis of the pipe may lie outside it.

It follows from these definitions that the wall of a pipe is a piecewise smooth $(n-1)$-dimensional surface, and that the pipe T lies in the $\frac{\eta}{2}$-neighbourhood of its own axis l (the latter follows from the definition of ζ_1).

We now construct a finite number of pipes $T_1, \ldots, T_s$, with walls $C_1, \ldots, C_s$ respectively, having the following properties:

a) $$D' \subset \bigcup_{i=1}^{s} (T_i \cup C_i)$$

and

b) $$T_i \cap T_j = 0 \quad \text{if} \quad i \neq j.$$

To do this, we take a finite ζ_2-net in D'. Let $x_1^0, \ldots, x_r^0$ be points of this ζ_2-net. We draw through each point x_i^0 an $(n-1)$-dimensional hyperplane π_i, orthogonal to the trajectory through this point. We denote by ω_i' the $(n-1)$-dimensional sphere in this hyperplane with centre at the point x_i^0 and radius $\frac{1}{2}\zeta_1$. We further denote by T_i' the pipe generated by ω_i', the wall of this pipe by C_i', and its axis by l_i'. ζ_2 is taken so that the ζ_2-neighbourhood in D' of the axis of the trajectory l_i' is contained in the pipe T_i'. Hence

$$D' \subset \bigcup_{i=1}^{r} (T_i' \cup C_i').$$

However, these are still not those pipes, which we require, since they intersect.

It would be possible to take all possible intersections of such pipes, if the number of components of such intersections were finite. However, we cannot assert this, a priori, and hence we must somewhat complicate the construction.

We put $T_1 = T_1'$ ($C_1 = C_1'$, $l_1 = l_1'$, correspondingly).

If $T_2' \cap T_1 = 0$, then we put $T_2 = T_2'$. If, however, $T_2' \cap T_1 \neq 0$, we proceed as follows. We consider the hyperplane π_2. We put $\tau_2 = T_1 \cap \pi_2$. The set τ_2 is an $(n-1)$-dimensional domain with a smooth boundary homeomorphic to an $(n-1)$-dimensional sphere. Also $\omega_2' \cap \tau_2 \neq 0$. We consider the difference

$$\omega_2' \setminus \bar{\tau}_2. \tag{8.12}$$

If this difference consists of a finite number k_2' of components, each of which has a piecewise smooth boundary, then we take them as $\omega_2, \ldots, \omega_{k_2'+1}$, and construct on them the pipes $T_2, \ldots, T_{k_2'+1}$, taking l_2' as the axis of each of these pipes, so that $l_i = l_2'$ ($i = 2, \ldots, k_2'+1$).

If the difference (8.12) is not of this form, then instead of ω_2' we take a larger $(n-1)$-dimensional domain $\omega_2'' \supset \omega_2'$ so that the boundary of ω_2'' is smooth, and ω_2'' is contained inside a sphere of radius ζ_1 with centre at the point x_2^0, and so that for ω_2'' the difference

$$\omega_2'' \setminus \bar{\tau}_2 \tag{8.13}$$

should now consist of a finite number k_2'' of components with piecewise smooth boundaries[1]. We take them as $\omega_2, \ldots, \omega_{k_2''+1}$ and construct the pipes $T_2, \ldots, T_{k_2''+1}$ as before.

$T_1 \cup C_1$ together with the pipes constructed at this step and their walls cover all that part of D', which was formerly covered by $T_1 \cup C_1$ and $T_2 \cup C_2$.

Suppose that we have constructed the pipes $T_1, \ldots, T_l$ covering together with their walls all that part of D', which is contained in the sum $\bigcup_{i=1}^{m} (T_i' \cup C_i')$. Let us consider the pipe T_{m+1}'. If $T_{m+1}' \cap \bigcup_{i=1}^{l} T_l = 0$, we take $T_{l+1} = T_{m+1}'$. If, however, $T_{m+1}' \cap \bigcup_{i=1}^{l} T_i \neq 0$, we then proceed as follows. We consider the hyperplane π_{m+1}, and put $\tau_{m+1} = \pi_{m+1} \cap \bigcup_{i=1}^{l} T_i$.

This is, in general, an open disconnected set. In the neighbourhood of ω_{m+1}', the boundary of the set τ_{m+1} is piecewise smooth. We take the difference

$$\omega_{m+1}' \setminus \bar{\tau}_{m+1}. \tag{8.14}$$

If this difference consists of a finite number k_{m+1}' of components, each of which has a piecewise smooth boundary, then we take them as $\omega_{l+1}, \ldots, \omega_{l+k_{m+1}'}$, and construct on them the pipes $T_{l+1}, \ldots, T_{l+k_{m+1}'}$, taking l_{m+1}' as the axis of each of these pipes ($l_i = l_{m+1}'$, $i = l+1, \ldots, l + k_{m+1}'$).

If the difference (8.14) is also not of this form, we then take instead of ω_{m+1}' a larger $(n-1)$-dimensional domain $\omega_{m+1}'' \supset \omega_{m+1}'$, so that the boundary of ω_{m+1}'' is smooth, and ω_{m+1}'' is contained inside the sphere of radius ζ_1 and centre at the point x_{m+1}^0, and so that for ω_{m+1}'' the difference

$$\omega_{m+1}'' \setminus \bar{\tau}_{m+1} \tag{8.15}$$

will now consist of a finite number k_{m+1}'' of components with piecewise smooth boundaries[2]. We take these as $\omega_{l+1}, \ldots, \omega_{l+k_{m+1}''}$, and construct the pipes $T_{l+1}, \ldots, T_{l+k_{m+1}''}$, as before.

Then

$$\bigcup_{i=1}^{l} (T_i \cup C_i) + T_{m+1}' + C_{m+1}' \subset \bigcup_{i=1}^{l+k_{m+1}''} (T_i \cup C_i),$$

and after r steps we construct the pipes $T_1, \ldots, T_s$, satisfying the conditions a) and b).

We term T_i a "through" pipe, if its axis l_i intersects each of the spheres $|x| = 1\frac{1}{8}$ and $|x| = 1\frac{7}{8}$. For each "through" pipe T_i we consider all segments of l_i between the spheres $|x| = 1\frac{1}{8}$ and $|x| = 1\frac{7}{8}$, and denote the k-th of them by $l_i^{(k)}$ $(k \geqslant 1)$.

We denote by $\tilde{l}_i^{(k)}$ the segment of $l_i^{(k)}$ between the spheres

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1 The existence of such a domain should be proved. The proof of this is not very difficult, but rather laborious. We leave it to the reader.

2 See above.

$|x| = 1\frac{1}{4}$ and $|x| = 1\frac{3}{4}$ (if there are several such segments, then we take an arbitrary one of them). Let $x_{i1}^{(k)}$ and $x_{i2}^{(k)}$ be the ends of $\tilde{l}_i^{(k)}$. We have

$$\int_{\tilde{l}_i^{(k)}} \frac{\partial f}{\partial l} dl = f(x_{i2}^{(k)}) - f(x_{i1}^{(k)})$$

($\frac{\partial f}{\partial l}$ is the derivative along the curve).

From inequality (8.10), it follows that the sign of $\frac{\partial f}{\partial l}$ is always constant on the trajectory, hence

$$\int_{\tilde{l}_i^{(k)}} \left| \frac{\partial f}{\partial l} \right| dl = |f(x_{i2}^{(k)}) - f(x_{i1}^{(k)})|,$$

and, from this same inequality (8.10), we obtain

$$\int_{\tilde{l}_i^{(k)}} |\operatorname{grad} f| \, dl < \frac{|f(x_{i2}^{(k)}) - f(x_{i1}^{(k)})|}{C(\alpha)}.$$

Or, noting that osc $f < 1$:

$$\int_{\tilde{l}_i^{(k)}} |\operatorname{grad} f| \, dl < \frac{1}{C(\alpha)}. \tag{8.16}$$

We denote by $E_i^{(k)}$ the set of points $x \in \tilde{l}_i^{(k)}$, at which

$$|\operatorname{grad} f| < \frac{4}{C(\alpha)}. \tag{8.17}$$

Since the length of $\tilde{l}_i^{(k)}$ is not less than $\frac{1}{2}$, then from (8.16) we get

$$\mu_1 E_i^{(k)} > \frac{1}{4}. \tag{8.18}$$

We draw through each point $x \in E_i^{(k)}$ an $(n-1)$-dimensional hyperplane $\pi_x^{(i,k)}$ orthogonal to $\tilde{l}_i^{(k)}$ at the point x, and denote by $v_x^{(i,k)}$ the component nearest to x of the intersection of $\pi_x^{(i,k)}$ with the pipe T_i. Since the pipe T_i lies in the $\frac{\eta}{2}$-neighbourhood of l_i, then, by the choice of η (conditions 2) and 4)) $v_x^{(i,k)}$ is situated in the η-neighbourhood of the point x. Hence, from the inequalities (8.11), (8.17), and because $|\operatorname{grad} f| > \beta$ everywhere in D'_δ, it follows that

$$|\operatorname{grad} f| \Big|_{v_x^{(i,k)}} < \frac{8}{C(\alpha)}. \tag{8.19}$$

We consider

$$\int_{\tilde{l}_i^{(k)}} \mu_{n-1} v_x^{(i,k)} \, dl.$$

As a consequence of conditions 2) and 4) for the choice of η, we obtain

$$\sum \int_{E_i^{(k)}} \mu_{n-1} v_x^{(i,k)} \, dl < 2\mu_n T_i,$$

and this means that, by (2.18), there exists a point $x_0^{(i,k)} \in E_i^k$, such that

$$\sum_k \mu_{n-1} v_{x_0^{(i,k)}}^{(i,k)} < 8\mu_n T_i. \tag{8.20}$$

We denote $\bigcup_k v_{x_0^{(i)}}^{(i)}$ by w_i. From (8.19) and (8.20), we find

$$\int_{w_i} |\operatorname{grad} f|\, d\sigma < \frac{64}{C(\alpha)} \mu_n T_i.$$

If T_i is not a "through" pipe, then we shall consider that w_i is empty.

We now put

$$\Sigma = \bigcup_{i=1}^{s} (C_i \cup w_i).$$

The set Σ, being the finite sum of piecewise smooth surfaces C_i and w_i, is a piecewise smooth surface. Besides, Σ separates in D the sphere $|x| = 1$ from the sphere $|x| = 2$. In fact, let l be a curve contained in D, and having limit points on both spheres $|x| = 1$ and $|x| = 2$. We suppose that $l \cap \bigcup_{i=1}^{s} C_i = 0$. Then the intersection $l \cap D'$ is contained in some one pipe T_i, and moreover, this pipe, evidently, is a "through" pipe, and hence, l_i certainly intersects w_i.

On the other hand

$$\int_\Sigma \left| \sum_{i,k=1}^{n} a_{ik}\gamma_i \frac{\partial f}{\partial x_k} \right| d\sigma =$$

$$= \int_{\bigcup_{i=1}^{s} w_i} \left| \sum_{i,k=1}^{n} a_{ik}\gamma \frac{\partial f}{\partial x_i} \right| d\sigma < n^2 \sum_{i=1}^{s} \int_{w_i} |\operatorname{grad} f|\, d\sigma < \frac{64n^2}{C(\alpha)} \sum_{i=1}^{s} \mu_n T_i$$

But $\sum_{i=1}^{s} \mu_i T_i$ does not exceed the volume of D'_δ. Hence

$$\int_\Sigma \left| \sum_{i,k=1}^{n} a_{ik}\gamma_i \frac{\partial f}{\partial x_k} \right| d\sigma < \frac{C}{M},$$

where the constant C depends on α and on n, which it was required to prove.

§9. Theorems on the growth and decay of a positive solution of a self-adjoint equation

The theorems of §§4-6 were all consequences of the principal lemma and the maximum principle, and consequently are valid under those hypotheses, for which the principal lemma (and the maximum principle) is valid. Since for equation (1.0.C) we proved the principal lemma (lemma 7.1) without any assumptions relative to the coefficients a_{ik} except the inequality (1.1)), it follows that for equation (1.0.C), the following theorem holds:

THEOREM 9.1. *Suppose that a sphere Q_R of arbitrary radius R with centre at the point O, contains a domain D including the point O and having limit points on the boundary of the sphere. Let Γ be that part of the boundary of the domain D, which lies inside Q_R. Let equation (1.0.C) be defined in D, and let*

$$\mu_n D = \sigma < \frac{\mu_n Q_R}{M},$$

where M is the constant of lemma 7.1.

Further, let $u(x)$ be a positive solution of equation (1.0.C) in D, continuous in $\overline{D}$, and vanishing on Γ. Then

$$u(0) < e^{-\frac{R^{\frac{1}{n-1}}}{C\sigma^{\frac{1}{n-1}}}} \max_{x \in D} u(x),$$

where C is a constant depending on the constant α of inequality (1.1), and on the dimensionality n of the space.

THEOREM 9.2 (Phragmén-Lindelöf type). *Let D be an unbounded domain, of the "type with solid angle size not larger than η", and let the equation (1.0.C) be defined in it. Further, let*

$$\eta < \frac{1}{2^n M},$$

where M is the constant of lemma 7.1.

Let there be determined in D a solution $u(x)$ of the equation, continuous in $\overline{D}$, and non-positive on the boundary of the domain D. Then, either 1) $u(x) \leqslant 0$ everywhere in D, or 2) if we put

$$M(R) = \sup_{\substack{|x|=R \\ x \in D}} u(x),$$

then

$$\liminf_{R \to \infty} \frac{M(R)}{R^{\frac{1}{K\eta^{\frac{1}{n-1}}}}} > 0,$$

where K is a constant, depending on the constant α of inequality (1.1) and on the dimensionality of the space.

THEOREM 9.3. *Let G be a domain lying outside some sphere Q with centre at the origin of coordinates. Let G be of the "type with solid angle size not larger than η". Let, further, Γ be that part of the boundary of the domain G, which lies strictly in the exterior of the sphere Q. Let equation (1.0.C) be defined in G, and η satisfy the inequality*

$$\eta < \frac{1}{2^n M}$$

(M is the constant of lemma 7.1).

Let there be defined in G a solution $u(x)$ of the equation, continuous *in G, positive inside the domain, and vanishing on* Γ.

Then, if $M(R)$ *has the same meaning as before, then, either*

$$\liminf_{R\to\infty} \left(M(R)/R^{K\eta^{n-1}}\right)^{\frac{1}{\frac{1}{n-1}}} > 0,$$

or

$$\liminf_{R\to\infty} \left(M(R)\cdot R^{K\eta^{n-1}}\right)^{\frac{1}{\frac{1}{n-1}}} < \infty,$$

where K is a constant, depending on the constant α *of inequality* (1.1), *and on the dimensionality n of the space.*

Chapter II

PROPERTIES OF SOLUTIONS WITH CHANGING SIGN

§0. Introduction

In this chapter we shall consider the equation

$$Lu \equiv \sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_k} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)\,u = 0, \qquad (1.0)^{1)}$$

concerning the coefficients we shall suppose that the a_{ik} are twice continuously differentiable, and the remaining coefficients are continuously differentiable, and these coefficients and their derivatives up to the order mentioned, inclusive, have moduli bounded by unity. Besides, we shall suppose that there are fulfilled the inequalities

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \geqslant \alpha\sum_{i=1}^{n}\xi_i^2, \qquad \alpha > 0 \qquad (1.1)$$

and

$$c \leqslant 0. \qquad (1.2)$$

It should be noted that for the principal results of this chapter, inequality (1.2) is not essential (boundedness of the moduli of the coefficients is sufficient). However, for the sake of simplifying the proofs, we shall require that it is satisfied.

Equation (1.0) *with these properties is denoted by* (1.0.*E*).

§ I. The uniqueness theorem

Latterly a large number of papers have been concerned with the problem of the uniqueness of the solution of Cauchy's problem for the elliptic equation of the form (1.0.E), or a more general one; we shall here be

1 We assign to formulae (1.0), (1.1), (1.2) in this chapter the same numbers as in chapter I.

concerned only with the equation of the form (1.0.E). The problem is posed thus:

In the domain D (fig. 3) a solution of equation (1.0.E) is determined, continuously differentiable up to the boundary. On the piece S of the boundary there is satisfied $u\,|_S = 0$, $\frac{\partial u}{\partial n}\Big|_S = 0$. To prove that $u \equiv 0$ everywhere in the domain D.

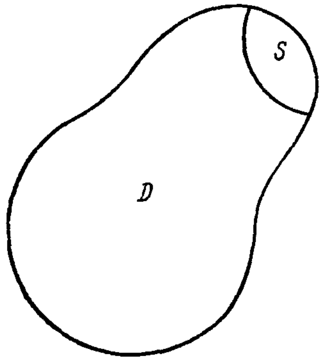

Fig. 3.

Using an observation of Hadamard [13], in proving the uniqueness theorem for Cauchy's problem for linear equations, we prove it by this means too for nonlinear equations.

In the case of analytic coefficients this uniqueness theorem is a consequence of a general theorem of Holmgren [14], proved by him in 1901: for equations of Kovalevsky type with analytic coefficients, the solution of Cauchy's problem is always unique in the class of sufficiently smooth functions (in the given case continuously differentiable).

In the case of nonanalytic coefficients the position was as follows.

For two independent variables (for general elliptic systems) the uniqueness of the solution of Cauchy's problem was proved by Carleman in 1933, [15], [38]. Then, for a long time there was no advance in this problem. After 20 years, beginning in 1953, there began to appear papers, in which the uniqueness of the solution of Cauchy's problem was established for the equation

$$\Delta u + \ldots = 0$$

with several independent variables (Douglis [16], Hartman and Wintner [17], Müller [18], Heinz [19]).

Beginning with 1956, there appeared a series of papers, in which the uniqueness theorem for the solution of Cauchy's problem was proved for equation (1.0.E), Aronszajn [20], [40], Cordes [21], Landis [22], M.M. Lavrent'ev [23].

The interest in the uniqueness of the solution of Cauchy's problem has not diminished since the series of papers on the uniqueness of Cauchy's problem continues (Calderon [24], Hörmander [25], [26]). An interesting analysis of the methods of proof was given in a seminar of Laurent Schwartz (it is published in the proceedings of the seminar in 1960).

The theorem of uniqueness of the solution of Cauchy's problem for elliptic equations of the second order is not a general fact, true for all systems of equations, as this holds in the case of analytic coefficients (theorem of Holmgren). For systems of equations, including elliptic (and also for one elliptic equation of higher order) there may not be uniqueness of the solution of Cauchy's problem. The first example of a system (parabolic), for which there is not uniqueness, was constructed in 1947 by Myshkis [35]. An example of such an elliptic system was given by Plis [36] in 1960.

Perhaps for equation (1.0.E), it would be more natural to speak not of the uniqueness theorem for the solution of Cauchy's problem, but of uniqueness theorems of the type of uniqueness theorems for analytic functions. The null conditions of Cauchy are somewhat strong. It is sufficient to demand that at an interior point of the domain the solution tends to zero faster than any power (Cordes [21]) or that on the boundary of the domain at some point x_0 the solution $u(x)$ and its normal derivative $\frac{\partial u(x)}{\partial n}$ tends to zero along the boundary faster than

$$e^{-\frac{1}{|x-x_0|^C}},$$

where C is some constant (Landis [22], M.M. Lavrent'ev [23]. The latter obtained a more precise value of the constant C: $2+\delta$, $\delta > 0$).

Another form of the uniqueness theorem gives the admissible rate of decay of the solution at infinity: let the solution $u(x)$ of equation (1.0.E) be determined in the cylinder unbounded on the right

$$\sum_{i\geqslant 2} x_i^2 < r^2, \qquad x_1 > 0,$$

and satisfy the inequality

$$|u| < e^{-e^{Cx_1}},$$

where C is some constant, depending on the equation and on r.

Then $u \equiv 0$ (Landis [22]).

We note that there exists a harmonic function not identically zero, decaying with speed e^{-e^x} (Re e^{-e^z} in the strip $0 < y < \pi$).

The problem of the continuous dependence of the solution of Cauchy's problem on the initial conditions is more interesting from the point of view of applications to other problems of the qualitative theory. An example by Hadamard is known, showing that, in the case of Cauchy's problem for Laplace's equation, there is no continuous dependence on the initial conditions in the usual sense.

However, it appears that in the class of uniformly bounded solutions there is a continuous dependence on the initial conditions (for Laplace's equation this was previously known to Carleman). The mechanism of this can be explained as follows. If we attempt to solve Cauchy's problem for Laplace's equation by Fourier's method, then we see that incorrectness arises on account of the high harmonics. But the high harmonics give exponential growth of velocity increasing with the number of harmonics. If we require the solution to be bounded on moving away somewhat from the hyperplane, on which the initial conditions are given, we arrive at the position where the coefficients fall in a geometric progression, as the number of harmonics is increased. This also guarantees the continuous dependence on the initial conditions, (it is, of course, in the class of those initial conditions, for which the coefficients decrease so quickly.

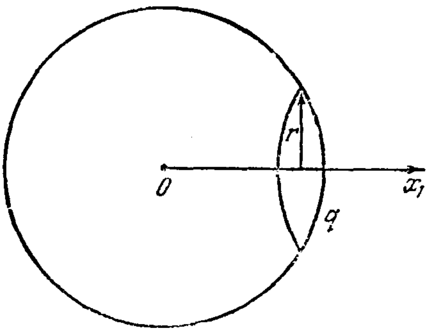

Fig. 4.

For equation (1.0.E), the theorem

on the continuous dependence on the initial conditions can be formulated as follows. Let Q (fig. 4) be the sphere $|x| \leqslant 1$, and q the part of its boundary defined by the inequality

$$\sum_{i\geqslant 2} x_i^2 < r^2.$$

Let the solution $u(x)$ of equation (1.0.E) be determined in Q, with modulus not exceeding unity. Let

$$\left| u \right|\Big|_q < \varepsilon, \qquad \left|\frac{\partial u}{\partial n}\right|\Big|_q < \varepsilon.$$

Then there holds the inequality

$$|u(0)| < \varepsilon^{r^C}.$$

(This was proved by me [22] for a sufficiently large constant C, depending on the equation; M.M. Lavrent'ev [23] by another different method found that for this constant one can take $2 + \delta$, $\delta > 0$, putting $\varepsilon^{C_1 r^{2+\delta}}$ in the right-hand side of the inequality, where C_1 depends on the equation.)

By a transformation of coordinates this theorem is extended to a domain of arbitrary form, and to all its interior points (the estimates are changed in a corresponding manner).

Another form of continuous dependence is given by the analogue of Hadamard's three circle theorem for analytic functions.

This theorem is formulated and proved in the next section. Later, (in §3) we shall obtain from this theorem the previously formulated theorem on the limiting velocity of decay of the solution in a cylinder (according to an iterated exponent).

An evident consequence of this theorem is the theorem concerning the identical vanishing of the solution, which decreases more quickly than any power on approaching an interior point of the domain, and a fortiori, the uniqueness theorem about the solution of Cauchy's problem.

It should be noted that the proof of the theorem of the next section is close, in its ideas, to the proof of the uniqueness of the solution of the problem of Cauchy-Heinz-Cordes.

§2. Three-sphere theorem[1]

We turn to the three circle theorem for analytic functions ([27], p. 469). It is formulated thus:

Let $f(z)$ be a function of a complex variable, defined in the annulus $r_1 \leqslant |z| \leqslant r_2$, analytic in the open ring $r_1 < |z| < r_2$, and continuous in the closed ring $r_1 \leqslant |z| \leqslant r_2$. We put

$$M(r) = \max_{|z|=r} |f(z)|.$$

Then, for any r ($r_1 < r < r_2$), we have the inequality

[1] A short account of this theorem is given in [41].

$$\ln M(r) \leqslant \ln M(r_1) \frac{\ln \frac{r}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M(r_2) \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}. \tag{2.1}$$

Let $r_2 = 1$, and $M(r_2) = 1$. Then, inequality (2.1) gives us

$$M(r) = M(r_1)\, e^{\frac{\ln r}{\ln r_1}}. \tag{2.2}$$

Let $M(r_1) = r_1^{\beta}$. Then, from inequality (2.2), we get

$$M(r) \leqslant r^{\beta}. \tag{2.3}$$

The inequality (2.3) is equivalent to Hadamard's theorem. In fact, in this form it will be convenient for us to transfer it to the solutions of elliptic equations.

We prove the following statement.

THEOREM 2.1. *In the sphere Q_1 of radius unity, let there be determined a solution $u(x)$ of equation* (1.0.E), *which is continuous in the closed sphere. We denote*

$$M(r) = \max_{|x|=r} |u(x)| \qquad (0 < r \leqslant 1).$$

Let

$$M(1) = 1. \tag{2.4}$$

Let, for any r_1 $(0 < r_1 < 1)$,

$$M(r_1) = r_1^{\beta}. \tag{2.5}$$

Then, for any r $(r_1 < r < 1)$, we have

$$M(r) \leqslant (Cr)^{\beta} \ln \frac{C}{r}, \tag{2.6}$$

where C is a constant depending on the constant α of inequality (1.1) *and on the dimensionality n of the space.*

(The presence of the factor $\ln C/r$, distinguishing the inequality (2.6) from inequality (2.3), is connected, possibly, with the method of proof).

Theorem 2.1 can be reformulated in a form closer to the classical theorem of Hadamard.

THEOREM 2.2. *In a sphere Q_{r_2} of radius $r_2 \leqslant 1$, let there be determined a solution $u(x)$ of equation* (1.0.E), *continuous in the closed sphere, and $M(r)$ $(0 < r \leqslant r_2)$ have the same meaning as before.*

Then, for any r_1 and r $(0 < r_1 < r < r_2)$, there holds the inequality

$$\ln M(r) \leqslant \ln M(r_1) \frac{\ln \frac{Cr}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M(r_2) \frac{\ln \frac{Cr}{r_1}}{\ln \frac{r_2}{r_1}} + \ln \ln \frac{Cr_2}{r}, \tag{2.7}$$

where C is a constant depending on α of inequality (1.1), *and on n.*

We first show how from theorem 2.1 we obtain theorem 2.2, and then proceed to the proof of theorem 2.1.

We suppose that theorem 2.1 is valid, and let $u(x)$ be a solution of equation (1.0.E) in the sphere $|x| \leqslant 1$. We put

$$u_1(x) = \frac{u(r_2 x)}{M(r_2)}. \qquad (2.8)$$

The function $u_1(x)$ is defined in the unit sphere and $\max_{x=1} |u_1(x)| = 1$. Further, $u_1(x)$ satisfies the equation

$$\sum_{i,k=1}^{n} a_{ik}(r_2 x) \frac{\partial^2 u_1}{\partial x_i \partial x_k} + r_2 \sum_{i=1}^{n} b_i(r_2 x) \frac{\partial u_1}{\partial x_i} + r_2^2 C(r_2 x) u_1 = 0.$$

We see that this equation satisfies the same inequalities as the initial equation (since $r_2 < 1$, and the derivatives of the coefficients of this equation have moduli less than the derivatives of the corresponding coefficients of the initial equation).

We put

$$M_1(r) = \max_{|x|=r} |u_1(x)|,$$

and let $M_1(r_1/r_2) = (r_1/r_2)^{\beta}$, i.e. $\beta = \ln M_1(r_1/r_2)/\ln(r_1/r_2)$. Then, from inequality (2.6), we get

$$M_1\left(\frac{r}{r_2}\right) \leqslant \left(\frac{Cr}{r_2}\right)^{\frac{\ln M_1\left(\frac{r_1}{r_2}\right)}{\ln \frac{r_1}{r_2}}} \ln \frac{Cr_2}{r} = M_1\left(\frac{r_1}{r_2}\right)^{\frac{\ln \frac{Cr}{r_2}}{\ln \frac{r_1}{r_2}}} \ln \frac{Cr_2}{r},$$

or

$$\frac{M(r)}{M(r_2)} \leqslant \left(\frac{M(r_1)}{M(r_2)}\right)^{\frac{\ln \frac{Cr}{r_2}}{\ln \frac{r_1}{r_2}}} \ln \frac{Cr_2}{r},$$

whence, in turn,

$$\ln M(r) \leqslant \ln M(r_1) \frac{\ln \frac{Cr}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M(r_2) \frac{\ln \frac{Cr}{r_1}}{\ln \frac{r_2}{r_1}} + \ln \ln \frac{Cr_2}{r}.$$

We proceed now to the proof of theorem 2.1.

1°. We use the following result of Cordes (cf. [21], pp. 246-254). Let there be defined in the sphere $|x| \leqslant 1$ the self-adjoint operator

$$L_0 \equiv \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x) \frac{\partial}{\partial x_k}\right),$$

the coefficients of which are twice continuously differentiable, and satisfy the inequalities

$$|a_{ik}| < 1, \quad \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k > \alpha \sum_{i=1}^{n} \xi_i^2, \quad \alpha > 0, \qquad (2.9)$$

$$\left|\frac{\partial a_{ik}}{\partial x_j}\right| < 1, \quad \left|\frac{\partial^2 a_{ik}}{\partial x_j \partial x_l}\right| < 1.$$

Then, there exists a transformation of coordinates

$$\left.\begin{array}{l} y_1 = y_1(x_1, \ldots, x_n), \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ y_n = y_n(x_1, \ldots, x_n), \end{array}\right\} \tag{2.10}$$

having the properties:

1) The transformation (2.10) leaves the origin of coordinates unchanged, and transforms the sphere $|x| < 1$ into a domain D, containing the sphere $|y| \leqslant C_1$.

2) The transformation (2.10) is continuously differentiable and its Jacobian $D(y)/D(x)$ satisfies the inequality

$$0 < C_2 < \frac{D(y)}{D(x)} < C_3, \tag{2.11}$$

the derivatives $\frac{\partial y_i}{\partial x_k}$ and $\frac{\partial x_i}{\partial y_k}$ satisfy the inequalities

$$\left|\frac{\partial y_i}{\partial x_k}\right| < C_4, \quad \left|\frac{\partial x_i}{\partial y_k}\right| < C_5 \qquad (i, k = 1, \ldots, n). \tag{2.12}$$

3) The transformation (2.10) is twice continuously differentiable, except, perhaps, at the origin, and the following inequalities hold:

$$\left|\frac{\partial^2 y_i}{\partial x_k\, \partial x_l}\right| < \frac{C_6}{|y|}, \quad \left|\frac{\partial^2 x_i}{\partial y_k\, \partial y_l}\right| < \frac{C_7}{|y|}. \tag{2.13}$$

4) We introduce in the space y, polar coordinates ρ, η ($\rho = |y|$ and η is the point $\left(\frac{y_1}{|y|}, \ldots, \frac{y_n}{|y|}\right)$ on the unit sphere $K_1 = \{|y| = 1\}$). Then the operator L_0 in these polar coordinates has the form

$$L_0 \equiv \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial}{\partial x_k}\right) \equiv$$

$$\equiv \frac{\partial^2}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2} M_\rho + \sum_{i=1}^{n} p_i(x)\frac{\partial}{\partial x_i}, \tag{2.14}$$

where

$$|p_i| < C_8 \tag{2.15}$$

and M_ρ, for each fixed ρ $(0 < \rho \leqslant C_1)$, is an elliptic operator on the unit sphere K_1, having the properties:

a) for each pair of functions ω_1 and ω_2, defined and twice continuously differentiable on the sphere K_1, we have

$$\int_{K_1} \omega_1 M_\rho \omega_2\, d\sigma_1 = \int_{K_1} \omega_2 M_\rho \omega_1\, d\sigma_1; \tag{2.16}$$

b) for any function ω, defined and twice continuously differentiable on the sphere K_1, we have the inequality

$$\frac{d}{d\rho}\int_{K_1} \omega M_\rho \omega\, d\sigma_1 \leqslant 0 \tag{2.17}$$

(on (2.16) and (2.17) $d\sigma_1$ denotes the element of surface of the unit sphere K_1).

The constants C_1 - C_8 depend on the constant α and on the dimensionality n of the space. In future, the letter C, supplied with suffices, will denote constants depending only on α and on n.

2°. We make the substitution (2.10) in equation (1.0.E). We get

$$Lu \equiv \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) + \sum_{k=1}^{n}\left(-\sum_{i=1}^{n}\frac{\partial a_{ik}}{\partial x_i} + b_k\right)\frac{\partial u}{\partial x_k} + cu \equiv L_1 u + L_2 u, \quad (2.18)$$

where

$$L_1 \equiv \frac{\partial^2}{\partial \varrho^2} + \frac{n-1}{\varrho}\frac{\partial}{\partial \varrho} + \frac{1}{\varrho^2} M_\varrho$$

and

$$L_2 \equiv \sum_{k=1}^{n}\left(-\sum_{i=1}^{n}\frac{\partial a_{ik}}{\partial x_i} + b_k + p_k\right)\frac{\partial}{\partial x_k} + c \equiv \sum_{k=1}^{n} d_k \frac{\partial}{\partial x_k} + c.$$

Then, in view of (2.12), (2.15),

$$|d_k| < C_9. \quad (2.19)$$

Our solution $u(y)$, being defined in the domain D of the space y, is defined in the sphere $|y| < C_1$, and satisfies in it, by inequality (2.4), the inequality

$$|u(y)| < 1. \quad (2.20)$$

Also, by (2.12), we have $|y| < C_{10}|x|$.

We make in addition a similarity transformation of the space y, with coefficient of similarity equal to $2/C_{10}$. The new variables are for the sake of simplicity denoted by the same letters y, but the radius of the sphere in which they are changed is, as before, denoted by C_1. Then, if L_1 in the new variables has the same form as previously,

$$L \equiv \frac{4}{C_{10}^2} L_1 + L_2 \quad \text{and} \quad |y| > 2|x|.$$

Hence, for the sphere $|y| < C_{10} r_1$, in accordance with (2.5), we have the inequality

$$|u(y)| \leqslant r_1^\beta,$$

or, putting $2r_1 = \rho_1$, we get

$$|u(y)|\big|_{|y|=\varrho_1} \leqslant \left(\frac{\varrho_1}{2}\right)^\beta \quad (2.21)$$

From the inequality (2.21), in view of Bernstein's inequality [28], we get

$$\sum_{i=1}^{n} \left|\frac{\partial u}{\partial y_i}\right|\Bigg|_{|y|\leqslant \frac{\varrho_1}{2}} \leqslant C_{11}\left(\frac{\varrho_1}{2}\right)^{\beta-1} \quad (2.22)$$

and

$$\sum_{i,k=1}^{n} \left|\frac{\partial^2 u}{\partial y_i\,\partial y_k}\right|\Bigg|_{|y|\leqslant \frac{\varrho_1}{2}} \leqslant C_{12}\left(\frac{\varrho_1}{2}\right)^{\beta-2} \quad (2.23)$$

The inequality (2.20) together with Bernstein's inequality gives

$$\sum_{i=1}^{n} \left| \frac{\partial u}{\partial y_i} \right\|_{|y| \leq \frac{C_1}{2}} \leq C_{13} \tag{2.24}$$

and

$$\sum_{i,\,k=1}^{n} \left| \frac{\partial^2 u}{\partial y_i\, \partial y_k} \right\|_{|y| \leq \frac{C_1}{2}} \leq C_{14}. \tag{2.25}$$

3^0. We select an arbitrary η $(0 < \eta < (\frac{1}{2}\rho_1)^{2\beta})$, and approximate to $u(y)$ be a thrice continuously differentiable function $v(y)$, so that

$$\max_{\frac{\varrho_1}{2} \leq |y| \leq C_1} \left(|v(y) - u(y)| + \sum_{i=1}^{n} \left| \frac{\partial v}{\partial y_i} - \frac{\partial u}{\partial y_i} \right| + \sum_{i,\,k=1}^{n} \left| \frac{\partial^2 v}{\partial y_i\, \partial y_k} - \frac{\partial^2 u}{\partial y_i\, \partial y_k} \right| \right) < \eta. \tag{2.26}$$

Let $f(\rho)$ be some fixed thrice continuously differentiable function, defined on the interval $\left[\frac{1}{2}C_1,\ \frac{3}{4}C_1\right]$ and having the properties:

$$0 \leq f(\varrho) \leq 1, \quad f\left(\frac{1}{2}C_1\right) = 1, \quad f\left(\frac{3}{4}C_1\right) = 0,$$

$$f'\left(\frac{1}{2}C_1\right) = f''\left(\frac{1}{2}C_1\right) = f'''\left(\frac{1}{2}C_1\right) = f'\left(\frac{3}{4}C_1\right) = f''\left(\frac{3}{4}C_1\right) = f'''\left(\frac{3}{4}C_1\right) = 0.$$

We shall suppose that $r_1 < 1/C_{10}$, and we fix some number $\rho_0 < \frac{1}{2}$, such that

$$\varrho_1 < \varrho_0 < C_1, \tag{2.27}$$

and put

$$w(y) = \begin{cases} v(y) & \text{if } \frac{1}{2}\varrho_1 \leq |y| \leq \frac{1}{2}\varrho_0, \\ v(y)\, f\left(\frac{C_1}{\varrho_0}|y|\right) & \text{if } \frac{1}{2}\varrho_0 < |y| \leq \frac{3}{4}\varrho_0, \\ 0 & \text{if } \frac{3}{4}\varrho_0 < |y| \leq \varrho_0. \end{cases}$$

Then for the function $w(y)$ we have

$$|Lw(y)| \leq C_{15}\eta \quad \text{if} \quad \frac{\varrho_1}{2} < |y| < \frac{\varrho_0}{2}, \tag{2.28}$$

$$|Lw(y)| \leq \frac{C_{16}}{\varrho_0^2} \quad \text{if} \quad \frac{\varrho_0}{2} < |y| < C_1, \tag{2.29}$$

$$|w(y)\|_{|y| = \frac{\varrho_1}{2}} \leq \left(\frac{\varrho_1}{2}\right)^{\beta} + \eta, \tag{2.30}$$

$$\sum_{i=1}^{n} \left| \frac{\partial w}{\partial y_i} \right\|_{|y| = \frac{\varrho_1}{2}} \leq C_{11} \left(\frac{\varrho_1}{2}\right)^{\beta-1} + \eta, \tag{2.31}$$

$$\sum_{i,\,k=1}^{n} \left| \frac{\partial^2 w}{\partial y_i\, \partial y_k} \right\|_{|y| = \frac{\varrho_1}{2}} \leq C_{12} \left(\frac{\varrho_1}{2}\right)^{\beta-2} + \eta. \tag{2.32}$$

4°. We put

$$z(y) = \frac{w(y)}{|y|^{\beta}} . \tag{2.33}$$

We have

$$L_1 w = L_1(\varrho^{\beta} z) = \frac{\partial^2}{\partial \varrho^2}(\varrho^{\beta} z) + \frac{n-1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho^{\beta} z) + \frac{1}{\varrho^2} M_{\varrho}(\varrho^{\beta} z) =$$

$$= \varrho^{\beta-2}\left[\varrho \frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial z}{\partial \varrho}\right) + (2\beta + n - 2)\varrho \frac{\partial z}{\partial \varrho} + \beta(\beta + n - 2) z + M_{\varrho} z\right] . \tag{2.34}$$

We put

$$2\beta + n - 2 = A \text{ and } \beta(\beta + n - 2) = B. \tag{2.35}$$

Then

$$\frac{(L_1 w)^2}{\varrho^{2\beta-4}} = \left[\varrho \frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial z}{\partial \varrho}\right) + Bz + M_{\varrho} z\right]^2 + A^2 \varrho^2 \left(\frac{\partial z}{\partial \varrho}\right)^2 +$$

$$+ 2A\left[\varrho \frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial z}{\partial \varrho}\right)\varrho \frac{\partial z}{\partial \varrho} + B\varrho z \frac{\partial z}{\partial \varrho} + \varrho \frac{\partial z}{\partial \varrho} M_{\varrho} z\right] .$$

or

$$\frac{(L_1 w)^2}{\varrho^{2\beta-3}} \geqslant A^2 \varrho \left(\frac{\partial z}{\partial \varrho}\right)^2 + A\left[\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial z}{\partial \varrho}\right)^2 + B \frac{\partial z^2}{\partial \varrho} + 2 \frac{\partial z}{\partial \varrho} M_{\varrho} z\right] . \tag{2.36}$$

Multiplying the left and right-hand sides of inequality (2.36) by $1/\rho^{n-1}$ and integrating over the domain $\frac{1}{2}\rho_1 < |y| < \rho_0$, we get

$$\int\limits_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{(L_1 w)^2 \, dy}{\varrho^{2\beta+n-4}} \geqslant$$

$$\geqslant \int\limits_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{A^2 \varrho \left(\frac{\partial z}{\partial \varrho}\right)^2 + A\left[\frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial z}{\partial \varrho}\right)^2 + B \frac{\partial z^2}{\partial \varrho} + 2 \frac{\partial z}{\partial \varrho} M_{\varrho} z\right]}{\varrho^{n-1}} \, dy.$$

or

$$\int\limits_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}} \, dy \geqslant A^2 \int\limits_{K_1} d\sigma_1 \int\limits_{\frac{\varrho_1}{2}}^{\varrho_0} \varrho \left(\frac{\partial z}{\partial \varrho}\right)^2 d\varrho +$$

$$+ A \int\limits_{K_1} d\sigma_1 \int\limits_{\frac{\varrho_1}{2}}^{\varrho_0} \frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial z}{\partial \varrho}\right)^2 d\varrho + AB \int\limits_{K_1} d\sigma_1 \int\limits_{\frac{\varrho_1}{2}}^{\varrho_0} \frac{\partial z^2}{\partial \varrho} d\varrho +$$

$$+ 2A \int\limits_{\frac{\varrho_1}{2}}^{\varrho_0} d\varrho \int\limits_{K_1} \frac{\partial z}{\partial \varrho} M_{\varrho} z \, d\sigma. \tag{2.37}$$

We consider the last of the integrals on the right-hand side of (2.37). For an arbitrary point $\tilde{\rho}$, $\frac{1}{2}\rho_1 < \tilde{\rho} < \rho_0$, we have

$$\frac{d}{d\varrho}\int_{K_1} zM_\varrho z\, d\sigma_1\,|_{\varrho=\widetilde{\varrho}} = \frac{d}{d\varrho}\int_{K_1} z\,|_{\varrho=\widetilde{\varrho}} M_\varrho (z\,|_{\varrho=\widetilde{\varrho}})\, d\sigma + \\ + \int_K \frac{\partial z}{\partial\varrho} M_\varrho z\, d\sigma_1\,|_{\varrho=\widetilde{\varrho}} + \int_{K_1} zM_\varrho \frac{\partial z}{\partial\varrho}\, d\sigma_1\,|_{\varrho=\widetilde{\varrho}}.$$

Because of (2.16) and (2.17), the first of the integrals on the right is non-positive, and the second and third are equal. Hence

$$\int_{\frac{\varrho_1}{2}}^{\varrho_0} d\varrho \int_{K_1} \frac{\partial z}{\partial\varrho} M_\varrho z\, d\sigma \geqslant \frac{1}{2}\int_{\frac{\varrho_1}{2}}^{\varrho_0} \frac{d}{d\varrho}\left(\int_{K_1} zM_\varrho z\, d\sigma_1\right) d\varrho. \tag{2.38}$$

Combining (2.38) with (2.37), and noting that, when $|y| = \rho_0$ the function z together with its first and second order partial derivatives vanishes, we find

$$\int_{K_1} d\sigma_1 \int_{\frac{\varrho_1}{2}}^{\varrho_0} \varrho\left(\frac{\partial z}{\partial\varrho}\right)^2 d\sigma \leqslant \frac{1}{A^2}\int_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(Lw)^2}{\varrho^{2\beta+n-4}}\, dy + \\ + \frac{1}{A}\int_{K_1}\left(\varrho\frac{\partial z}{\partial\varrho}\right)^2\Big|_{\varrho=\frac{\varrho_1}{2}} d\sigma_1 + \frac{B}{A}\int_{K_1} z^2\Big|_{\varrho=\frac{\varrho_1}{2}} d\sigma_1 + \frac{2}{A}\int_{K_1} |zM_\varrho z|\Big|_{\varrho=\frac{\varrho_1}{2}} d\sigma_1. \tag{2.39}$$

However, by (2.30) and (2.33), and considering the choice of η,

$$|z|\,\Big|_{|y|=\frac{\varrho_1}{2}} \leqslant C_{17}. \tag{2.40}$$

Further, from (2.33)

$$\frac{\partial z}{\partial\varrho} = \frac{1}{\varrho^\beta}\frac{\partial w}{\partial\varrho} - \beta\frac{w}{\varrho^{\beta+1}},$$

and consequently from (2.30) and (2.31)

$$\varrho\left|\frac{\partial z}{\partial\varrho}\right|_{|y|=\frac{\varrho_1}{2}} \leqslant C_{18}. \tag{2.41}$$

We still have to evaluate $M_\rho z$. But $M_\rho z = \frac{1}{\rho^\beta} M_\rho w$, $M_\rho w$ is near to $M_\rho u$, and we can evaluate the latter from the equation. We have

$$M_\varrho u = -\varrho^2\left[\frac{\partial^2 u}{\partial\varrho^2} + \frac{n-1}{\varrho}\frac{\partial u}{\partial\varrho} + L_2 u\right].$$

Whence, with the help of inequalities (2.21), (2.22) and (2.23), we get

$$|M_\varrho u|\,\Big|_{\varrho=\frac{\varrho_1}{2}} \leqslant C_{19}\left(\frac{\varrho_1}{2}\right)^\beta.$$

Whence, in turn, considering (2.26) and the choice of η, we find

$$|M_\varrho z|\,\Big|_{\varrho=\frac{\varrho_1}{2}} \leqslant C_{20}. \tag{2.42}$$

We suppose that

$$\beta \geqslant 1. \tag{2.43}$$

Then (2.39) together with (2.40), (2.41), (2.42) and (2.35) gives us the inequality

$$\int_{K_1} d\sigma_1 \int_{\frac{\varrho_1}{2}}^{\varrho_0} \varrho \left(\frac{\partial z}{\partial \varrho}\right)^2 d\varrho \leqslant \frac{C_{21}}{\beta^2} \int_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}} dy + C_{22}\beta. \qquad (2.44)$$

We prove the following inequality:

$$\int_{\frac{\varrho_1}{2}}^{\varrho_0} \varrho \left(\frac{\partial z}{\partial \varrho}\right)^2 d\varrho \geqslant \frac{1}{4} \int_{\frac{\varrho_1}{2}}^{\varrho_0} \frac{z^2}{\varrho \ln^2 \frac{1}{\varrho}} d\varrho - C_{23} \qquad (2.45)$$

For this we put

$$z = \varphi \sqrt{\ln \frac{1}{\varrho}}$$

Then

$$\left(\frac{\partial z}{\partial \varrho}\right)^2 = \frac{1}{4} \frac{\varphi^2}{\varrho^2 \ln \frac{1}{\varrho}} + \left(\frac{\partial \varphi}{\partial \varrho}\right)^2 \ln \frac{1}{\varrho} - \frac{1}{2\varrho} \frac{\partial \varphi^2}{\partial \varrho},$$

which gives us

$$\int_{\frac{\varrho_1}{2}}^{\varrho_0} \varrho \left(\frac{\partial z}{\partial \varrho}\right)^2 d\varrho \geqslant \frac{1}{4} \int_{\frac{\varrho_1}{2}}^{\varrho_0} \frac{z^2 d\varrho}{\varrho \ln^2 \frac{1}{\varrho}} - \frac{1}{2} \frac{z^2}{\ln \frac{1}{\varrho}} \Bigg|_{\frac{\varrho_1}{2}}^{\varrho_0} \geqslant \frac{1}{4} \int_{\frac{\varrho_1}{2}}^{\varrho_0} \frac{z^2 d\varrho}{\varrho \ln^2 \frac{1}{\varrho}} - C_{23}.$$

Substituting (2.45) in (2.44), and transforming from z to w, we find

$$\int_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{w^2}{\varrho^{2\beta+n} \ln^2 \frac{1}{\varrho}} dy \leqslant \frac{C_{21}}{\beta^2} \int_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{(L_1 w)^2 dy}{\varrho^{2\beta+n-4}} + C_{24}\beta. \qquad (2.46)$$

5°. We now estimate

$$\int_{\frac{\varrho_1}{2} < |y| < \varrho_0} \frac{|\operatorname{grad} w|^2}{\varrho^{2\beta+n-3}} dy. \qquad (2.47)$$

For this we remember that by (2.14)

$$\frac{4}{C_{10}^2} L_1 = \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik} \frac{\partial}{\partial x_k}\right) - \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i}.$$

Therefore

$$L_1(w^2) = 2wL_1 w + \frac{C_{10}^2}{2} \sum_{i,k=1}^{n} a_{ik} \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_k},$$

and since

$$\sum_{i,k=1}^{n} a_{ik} \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_k} \geqslant \alpha |\operatorname{grad} w|^2,$$

then

$$|\operatorname{grad} w|^2 \leqslant \frac{2}{\alpha l_{10}^2} L_1(w^2) - \frac{4}{\alpha l_{10}^2} w L_1 w. \qquad (2.48)$$

Further

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} L_1(w^2)\frac{dy}{\varrho^{2\beta+n-3}} = \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} w^2 L_1\left(\frac{1}{\varrho^{2\beta+n-3}}\right)dy +$$

$$+\frac{2}{\left(\frac{\varrho_1}{2}\right)^{2\beta+n-3}}\int\limits_{|y|=\frac{\varrho_1}{2}} w\frac{\partial w}{\partial\varrho}\,d\sigma + \frac{(n)}{\left(\frac{\varrho_1}{2}\right)^{2\beta+n-2}}\int\limits_{|y|=\frac{\varrho_1}{2}} w^2 d\sigma,$$

which, with (2.48), gives

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{|\operatorname{grad} w|^2}{\varrho^{2\beta+n-3}}\,dy \leqslant \frac{2}{\alpha C_{10}^2}\left\{\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} w^2 L_1\left(\frac{1}{\varrho^{2\beta+n-3}}\right)dy +\right.$$

$$+\frac{2}{\alpha\left(\frac{\varrho_1}{2}\right)^{2\beta+n-3}}\int\limits_{|y|=\frac{\varrho_1}{2}} w\frac{\partial w}{\partial\varrho}\,d\sigma +$$

$$\left.+\frac{n-1}{\alpha\left(\frac{\varrho_1}{2}\right)^{2\beta+n-2}}\int\limits_{|y|=\frac{\varrho_1}{2}} w^2 d\sigma - \frac{2}{\alpha}\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0}\frac{wL_1w}{\varrho^{2\beta+n-3}}\,dy\right\}. \qquad (2.49)$$

We have

$$L_1\left(\frac{1}{\varrho^{2\beta+n-3}}\right) = \frac{(2\beta-1)(2\beta+n-3)}{\varrho^{2\beta+n-1}}. \qquad (2.50)$$

Then

$$\int\limits_{|y|=\frac{\varrho_1}{2}} w\frac{\partial w}{\partial\varrho}\,d\sigma \leqslant \left[\int\limits_{|y|=\frac{\varrho_1}{2}} w^2\,d\sigma \int\limits_{|y|=\frac{\varrho_1}{2}}\left(\frac{\partial w}{\partial\varrho}\right)^2 d\sigma\right]^{1/2},$$

and hence, by inequalities (2.21), (2.22), (2.26) and by the choice of η,

$$\int\limits_{|y|=\frac{\varrho_1}{2}} w\frac{\partial w}{\partial\varrho}\,d\sigma \leqslant C_{25}\left(\frac{\varrho_1}{2}\right)^{2\beta+n-2}. \qquad (2.51)$$

Finally

$$\left|\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0}\frac{wL_1w}{\varrho^{2\beta+n-3}}\,dy\right| = \left|\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0}\frac{w}{\varrho^{\beta+\frac{n}{2}-1}}\,\frac{L_1w}{\varrho^{\beta+\frac{n}{2}-2}}\,dy\right| \leqslant$$

$$\leqslant\left[\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0}\frac{w^2}{\varrho^{2\beta+n-2}}\,dy\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0}\frac{(L_1w)^2}{\varrho^{2\beta+n-4}}\,dy\right]^{1/2}. \qquad (2.52)$$

Combining now (2.49) with (2.50), (2.51), (2.52) and (2.21), we get

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{|\operatorname{grad} w|^2}{\varrho^{2\beta+n-3}}\,dy \leqslant C_{27}\beta^2 \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n-1}}\,dy +$$

$$+C_{28}\Bigg[\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n-2}}\,dy \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}}\,dy\Bigg]^{1/2} + C_{29} \leqslant$$

$$\leqslant \varrho_0\Bigg\{C_{27}\beta^2 \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n}}\,dy + C_{28}\times$$

$$\times\Bigg[\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n}}\,dy \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}}\,dy\Bigg]^{1/2}\Bigg\} + C_{29}.$$

Together with (2.46) this gives

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{|\operatorname{grad} w|^2}{\varrho^{2\beta+n-3}}\,dy \leqslant C_{30}\varrho_0 \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}}\,dy + C_{31}\beta^3. \tag{2.53}$$

6°. We have

$$\frac{4}{C_{10}^2} L_1 w = Lw - L_2 w = Lw - \sum_{i=1}^{n} d_i \frac{\partial w}{\partial y_i} - cw.$$

We put

$$n\cdot\max|d_i| = C_{32}.$$

Then, remembering that $|c| \leqslant 1$, we get

$$\frac{16}{C_{10}^4}(L_1 w)^2 \leqslant 3(Lw)^2 + 3C_{32}^2|\operatorname{grad} w|^2 + 3w^2. \tag{2.54}$$

From the inequality (2.46) it follows that

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n-4}}\,dy \leqslant C_{33}\varrho_0 \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}} + C_{34}\beta. \tag{2.55}$$

Let ρ_0 satisfy the inequality

$$\varrho_0 \leqslant \frac{C_{10}^4}{96(C_{30}C_{32}^2 + C_{33})}. \tag{2.56}$$

Then from inequalities (2.53), (2.54) and (2.55) we get

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(L_1 w)^2}{\varrho^{2\beta+n-4}}\,dy \leqslant 6 \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(Lw)^2}{\varrho^{2\beta+n-4}}\,dy + C_{35}\beta^3, \tag{2.57}$$

and, turning to (2.46), we can write

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n}\ln^2\frac{1}{\varrho}}\,dy \leqslant \frac{C_{36}}{\beta^2} \int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{(Lw)^2}{\varrho^{2\beta+n-4}}\,dy + C_{37}\beta. \tag{2.58}$$

We now recall inequalities (2.28) and (2.29). Estimating the right-hand side of (2.57) with their aid, we get

$$\int\limits_{\frac{\varrho_1}{2}<|y|<\varrho_0} \frac{w^2}{\varrho^{2\beta+n}\ln^2\frac{1}{\varrho}}\,dy \leqslant \frac{C_{38}}{\beta^2\varrho_0^{2\beta-2}} + C_{39}\beta. \tag{2.59}$$

7°. Let, now, $\tilde{\rho}$ ($\rho_1 < 2\tilde{\rho} < \rho_0$) be an arbitrary number. From the inequality (2.59), we find

$$\frac{1}{(2\tilde{\varrho})^n}\int\limits_{\frac{\varrho_1}{2}<|y|<2\tilde{\varrho}} w^2 dy \leqslant \left[C_{38}\left(\frac{2\tilde{\varrho}}{\varrho_0}\right)^{2\beta}\frac{\varrho_0^2}{\beta^2} + C_{39}\beta(2\tilde{\varrho})^{2\beta}\right]\ln^2\frac{1}{\tilde{\varrho}}$$

Since $\rho_0 < \frac{1}{2}$, then

$$\frac{1}{(2\tilde{\varrho})^n}\int\limits_{\frac{\varrho_1}{2}<|y|<2\tilde{\varrho}} w^2 dy \leqslant C_{40}\left(\frac{2\tilde{\varrho}}{\varrho_0}\right)^{2\beta}\ln^2\frac{1}{\tilde{\varrho}}.$$

But from this inequality it follows (cf. [29]) that

$$|w|\Big|_{|y|=\tilde{\varrho}} \leqslant C_{41}\left(\frac{2\tilde{\varrho}}{\varrho_0}\right)^{\beta}\ln\frac{1}{\tilde{\varrho}}.$$

We take now as ρ_0 the number

$$\min\left[\frac{1}{2},\ C_1,\ \frac{1}{6(C_{30}-C_{32}^2+C_{33})}\right] = C_{42}$$

(so that inequalities (2.27) and (2.26) are satisfied). Then, for any ρ ($\rho_1 < \frac{1}{2}C_{42}$) it is true that

$$|w|\Big|_{|y|=\varrho} \leqslant C_{41}\left(\frac{2\varrho}{C_{42}}\right)^{\beta}\ln\frac{1}{\varrho}.$$

Going from w and ρ to u and r, we get

$$|u|\Big|_{|x|=r} \leqslant C_{43}(C_{44}r)^{\beta}\ln\frac{C_{44}}{r} \leqslant (C_{45}r)^{\beta}\ln\frac{C_{44}}{r}$$

for all r ($C_{46}r_1 < r < C_{47}$). From this it follows that C_{48} exists, such that

$$|u|\Big|_{x=r} \leqslant (C_{48}r)^{\beta}\ln\frac{C_{48}}{r}$$

for r ($r_1 < r < 1$).

Thus, for $\beta \geqslant 1$, the theorem is proved. It remains to consider the case $\beta < 1$. There exists a constant C_{49}, such that

$$|\operatorname{grad} u|\Big|_{|x|\leqslant\frac{1}{2}} < C_{49}.$$

Hence

$$|u|\Big|_{|x|=r} \leqslant r_1^{\beta} + C_{49}(r-r_1) \quad \text{if} \quad r_1 < r < \frac{1}{2},$$

and, since $\beta < 1$, then

$$|u|\Big|_{|x|=r} \leqslant (C_{50}r)^{\beta}\ln\frac{C_{50}}{r} \quad \text{if} \quad r_1 < r < 1.$$

To complete the proof of the theorem, it is only necessary to put

$$C = \max(C_{48}, C_{50}).$$

§3. On the admissible rate of decay of the solution in a semicylinder

As an example of the application of the three spheres theorem, we shall obtain the following theorem.

THEOREM 3.1. *Let equation* (1.0.E) *be defined in the half cylinder* $U = \sum_{i=2}^{n} x_i^2 = r^2 \leqslant 1$, $x_1 \geqslant 0$. *There exists a constant* N, *depending on* r, *on the constant* α *of inequality* (1.1) *and on the dimensionality of space, such that whatever be the solution* $u(x)$ *of equation* (1.0.E), *determined in* U, *not identically zero, there holds the inequality*

$$\limsup \frac{M(x_1)}{e^{-e^{Nx_1}}} > 0, \tag{3.1}$$

$$M(\widetilde{x}_1) = \max_{x_1 = \widetilde{x}_1} |u(x_1, \ldots, x_n)|.$$

PROOF. It is evidently sufficient to consider the case where $|u(x)| < 1$. Let C be the constant of theorem 2.1, corresponding to the constant α of our equation.

We put

$$r_0 = \min\left[\left(\frac{r}{2C}\right)^2 \frac{r}{(C+1)e^{18}}\right]. \tag{3.2}$$

We denote by Q'_{x_1}, Q''_{x_1} and Q'''_{x_1} the spheres with centre at the point $(x_1, 0, \ldots, 0)$, and radii r_0, $2r_0$ and r respectively. We put

$$M'(x_1) = \sup_{x \in Q'_{x_1}} |u(x)| \text{ and } M''(x_1) = \sup_{x \in Q''_{x_1}} |u(x)|.$$

Let, for some $x_1 > r$

$$M'(x_1) = r_0^{\beta}, \quad \beta > 1. \tag{3.3}$$

Applying the theorem of the three spheres to the sphere Q''_{x_1}, we find, on considering (3.2)

$$M''(x_1) \leqslant r_0^{\frac{\beta}{2}} \ln \frac{Cr}{r_0} < r_0^{\frac{\beta}{3}}.$$

Since $Q'_{x_1 - r_0} \subset Q''_{x_1}$, then

$$M'(x_1 - r_0) \leqslant r_0^{\frac{\beta}{3}}.$$

Hence we find, by induction on k, that

$$M'(x_1 - kr_0) \leqslant r_0^{\frac{\beta}{3^k}} \tag{3.4}$$

$$-(k-1)r_0 > r \qquad \frac{\beta}{3^{k-1}} > 1.$$

provided that $x_1 - (k-1)r_0 > r$ and $\beta/3^{k-1} > 1$.

Let, now, u be an arbitrary solution of equation (1.0.E), not identically zero. Then, by the theorem of uniqueness of the solution of

Cauchy's problem, $M'(r) > 0$. Let

$$M'(r) = r_0^{\beta_0}, \quad \beta_0 > 1.$$

From inequality (3.4), we find that, for any $x_1 > r + r_0$, it is true that

$$M'\left(\left[\frac{x}{r_0}\right] r_0\right) \geqslant r_0^{\beta_0 \cdot 3^{\left[\frac{x}{r_0}\right]}} > e^{-e^{\frac{\ln 3}{r_0}(x_1+1) + \ln \beta_0 + \ln \ln \frac{1}{r_0}}},$$

whence, it follows that there exist arbitrarily large values of x_1, such that

$$M(x_1) > e^{-e^{\frac{2 \ln 3}{r_0} x_1}},$$

and since r_0 depends on r, α and on n, the dimensionality of space, the theorem is proved.

This theorem admits the following generalization.

THEOREM 3.2. *Let equation (1.0.E) be defined in some domain D. Let $|u| < M$ in D, and let $u(x) = a \neq 0$ at some point $x \in D$. Let y be some other point of the domain D, and let it be possible to join the point x to the point y by a sequence consisting of k spheres of radius r, contained in D, and such the centre of one sphere of the sequence is situated inside the sphere that precedes it, the centre of the first sphere coincides with x, and the centre of the last with y. Then, there exists a point z, such that*

$$|y - z| < r \text{ and } |u(z)| > \frac{1}{M}\left(\frac{a}{M}\right)^{e^{-Ck}},$$

where C is a constant, depending on r, the constant α of inequality (1.0.E), and on the dimensionality of the space.

The proof of this theorem differs little from the proof of the previous theorem, and we omit it.

§4. The relation between the number of changes of sign of a solution and its growth

In this section we shall consider the equation

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) = 0 \qquad (1.0.C)$$

(cf. p. 22). As in §7, chap. I, we shall mean by a solution of it a continuous function satisfying the integral identity (7.1) of chapter I.

Let a continuous function $u(x)$ be defined in the spherical layer $T = \{R_1 \leqslant x \leqslant R_2\}$. We denote by E^+ the set of points $x \in T$, where $u(x) > 0$, and by E^- the set of points $x \in T$, where $u(x) < 0$. We shall mean by a halfwave of amplitude not less than $m > 0$, any component D of the set $E = E^+ \cup E^-$, having limit points on each of the spheres $|x| = R_1$ and $|x| = R_2$, such that

$$\min_{R_1 \leqslant r \leqslant R_2} \max_{\substack{x \in D \\ |x|=r}} |u(x)| \geqslant m. \tag{4.1}$$

LEMMA 4.1. *Let the equation (1.0.C) be defined in the sphere* $Q_R = \{|x| \leqslant R\}$ *of arbitrary radius R. Let u(x) be a solution of it in* Q_R, *such that in the spherical layer* $T_R = \{\frac{1}{2}R \leqslant |x| \leqslant R\}$, *this solution has N halfwaves* $D_1, \ldots, D_N$ *and the amplitude of the i-th halfwave is not less than* $m_i > 0$.

Then, there exist not less than $\frac{1}{2}N$ *halfwaves* D_i, *such that*

$$\max_{x \in D_i} |u(x)| > m_i 2^{\frac{N^{\frac{1}{n-1}}}{C_1} - 1}, \tag{4.2}$$

where C_1 *is a constant depending on the constant* α *in the inequality, and on the dimensionality n of space.*

PROOF. Since the number of halfwaves is equal to N, and they do not intersect, there exist among them not less than $\frac{1}{2}N$ halfwaves $D_{j_1}, \ldots, D_{j_s}$, $s \geqslant \frac{1}{2}N$, such that

$$\mu_n D_{j_p} < \frac{2\omega_n R^n}{N} \quad (p = 1, 2, \ldots, s) \tag{4.3}$$

(ω_n is the volume of the n-dimensional unit sphere). We take some D_{j_p}. We shall, for the sake of definiteness, consider that $D_{j_p} \subset E^+$. In the contrary case, we change the sign of u.

Let M be the constant of lemma 7.1, chap. I, selected for that value of α, which enters into the given equation.

We suppose that

$$N > 2^{3n+1} M, \tag{4.4}$$

and let

$$k = \left[\left(\frac{N}{2^{2n+2} M} \right)^{\frac{1}{n-1}} \right]. \tag{4.5}$$

We denote by d_i the intersection of D_{j_p} with the spherical layer

$$\frac{R}{2} + \frac{i-1}{2k} R < |x| < \frac{R}{2} + \frac{i}{2k} R \qquad (i = 1, 2, \ldots, k).$$

In view of the inequalities (4.3), (4.4) and (4.5), there exist at least $\frac{1}{2}k$ different values of i, such that

$$\mu_n d_i < \frac{\omega_n \left(\frac{R}{4k} \right)^n}{M}. \tag{4.6}$$

Let these values of i be $i_1, i_2, \ldots, i_r$, $r \geqslant \frac{1}{2}k$. Let

$$m^{(l)} = \max_{\substack{|x| = \frac{R}{2} + \frac{i_l - \frac{1}{2}}{2k} \\ x \in d_i}} u(x) \qquad (l = 1, 2, \ldots, r),$$

and let this maximum be attained at the point $x^{(l)}$. We denote by $Q^{(l)}$ the sphere of radius $R/4k$, with centre at the point $x^{(l)}$, and by $g^{(l)}$ the component of the intersection $d_{i_l} \cap Q^{(l)}$, containing the point $x^{(l)}$. Since from (4.6) there follows a fortiori

$$\mu_n g^{(l)} < \frac{\mu_n Q^{(l)}}{M},$$

then, applying lemma 7.1, chap. I, we find

$$\sup_{x \in g^{(l)}} u(x) > 2m^{(l)}.$$

By the maximum principle

$$m^{(l+1)} \geqslant \sup_{x \in g^{(l)}} u(x),$$

and since $m^{(1)} \geqslant m_{j_p}$, then

$$\max_{x \in D_{j_p}} u(x) \geqslant \sup_{x \in g^{(r)}} u(x) > m_{j_p} 2^{\frac{K}{2}};$$

and since by (4.4) and (4.5)

$$K > \frac{N^{\frac{1}{n-1}}}{2^{\frac{2n+3}{n-1}} M^{\frac{1}{n-1}}},$$

then

$$\max_{x \in D_{j_p}} u(x) > m_{j_p} 2^{\frac{N^{\frac{1}{n-1}}}{2^{\frac{3n+2}{n-1}} M^{\frac{1}{n-1}}}} \tag{4.7}$$

If we now put $C_1 = 2^{\frac{3n+2}{n-1}} M^{\frac{1}{n-1}}$, then, when condition (4.4) is satisfied, in view of (4.6), we shall have

$$\max_{x \in D_{j_p}} u(x) > m_{j_p} 2^{\frac{N^{\frac{1}{n-1}}}{C_1}} > m_{j_p} 2^{\frac{N^{\frac{1}{n-1}}}{C_1} - 1},$$

if, however, (4.4) is not satisfied, then inequality (4.2) is satisfied trivially, and the lemma is proved.

THEOREM 4.1. *Let equation (1.0.C) be defined in all space. Let $u(x)$ be a solution of this equation, also defined in all space. Let E^+ be the set of points x of space, where $u(x) > 0$, and E^- be the set of points x of space, where $u(x) < 0$. Let N be the total number of components of the sets E^+ and E^-. Then*

$$\liminf_{r\to\infty} \frac{M(r)}{r^{\frac{N^{\frac{1}{n-1}}}{C}-1}} > 0,$$

where $M(r) = \max_{|x|=r} |u(x)|$ *and* C *is a constant depending on* α *of inequality* (1.1), *and on the dimensionality of the space.*

PROOF. We note firstly that, by the maximum principle, every component of the set E^+ or E^- is an unbounded set. Let R_0 be so large, that the sphere Q_{R_0}, of radius R_0 with centre at the origin of coordinates, intersects all N components of the sets E^+ and E^-.

Let $g_1, \ldots, g_N$ be these components of the sets E^+ and E^-, and let

$$m = \min_i \Big(\sup_{x\in(g_i\cap Q_{R_0})} |u(x)| \Big).$$

Then, by the maximum principle,

$$\max_{\substack{x\in g_i \\ |x|=r>R_0}} |u(x)| \geqslant m. \tag{4.8}$$

Let C_1 be the constant of lemma 4.1, corresponding to α of inequality (1.1) for the given equation.

Let $r_k = R_0 2^k$ $(k > 1)$. We fix some k, and consider the spherical layers

$$T_1 = \{R_0 \leqslant |x| \leqslant 2R_0\},$$
$$T_2 = \{2R_0 \leqslant |x| \leqslant 2^2 R_0\},$$
$$\cdots\cdots\cdots\cdots$$
$$T_k = \{2^{k-1}R_0 \leqslant |x| \leqslant 2^k R_0\}.$$

For each $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, k$, we put

$$m_{ij} = \min_{2^{j-1}R_0\leqslant r\leqslant 2^j R_0} \Big(\max_{\substack{x\in g_i\\ |x|=r}} |u(x)|\Big) \quad \text{and} \quad m_{i,k+1} = \max_{\substack{x\in g_i\\ |x|=R_0 2^k}} |u(x)|.$$

By inequality (4.8)

$$m_{i,1} \geqslant m. \tag{4.9}$$

Further, by lemma 4.1, and by the maximum principle for any j $(j = 1, \ldots, k)$ there exist not less than $\frac{1}{2}N$ values of i: $i : i_{j,1}, \ldots, i_{j,s}$, $s \geqslant \frac{1}{2}N$, such that

$$m_{i_{j,r}j+1} > m_{i_{j,r}j} \cdot 2^{\frac{N^{\frac{1}{n-1}}}{C_1}-1} \qquad (r = 1, 2, \ldots, s). \tag{4.10}$$

Whence, in turn, it follows that there exists i_0, and not less than $k/2$ values of $j : j_1, \ldots, j_q$, $q \geqslant \frac{1}{2}k$, such that

$$m_{i_0 j_p+1} > m_{i_0 j_p} \cdot 2^{\frac{N^{\frac{1}{n-1}}}{C_1}-1} \qquad (p = 1, 2, \ldots, q).$$

Whence

$$m_{i_0 j_p+1} > m_{i_0 j_0}\cdot 2^{\left(\frac{N^{\frac{1}{n-1}}}{C_1}-1\right)\frac{k}{2}},$$

which, remembering (4.5), gives

$$M(r_k) > m\cdot 2^{\frac{k}{2}\left(\frac{N^{\frac{1}{n-1}}}{C_1}-1\right)} = m\left(\frac{r_k}{R_0}\right)^{\frac{1}{2}\left(\frac{N^{\frac{1}{n-1}}}{C_1}-1\right)}$$

and, for sufficiently large k,

$$M(r_k) > m r_k^{\frac{N^{\frac{1}{n-1}}}{2C_1}-1}.$$

Whence, by the maximum principle, for sufficiently large k

$$M(r) > m r^{\frac{N^{\frac{1}{n-1}}}{4C_1}-1},$$

and, to complete the proof, it only remains to put $C = 4C_1$.

COROLLARY. *If the solution $u(x)$ of equation (1.0.C) is defined in all space, and the sets E^+ and E^- have infinitely many components, then $M(r) = \max_{|x|=r} |u(x)|$ grows, as $r \to \infty$, faster than any power.*

Instead of equation (1.0.C), it would have been possible to consider equation (1.0.B) (p. 14). Here, instead of lemma 7.1 chap I, we should use lemma 3.2, chap. II.

Received by the editors 6th. July 1962.

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Translated by P.G. Gormley.