Small Divisors and the NLSE

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MSRI, September 2015

Classical Small Divisor Problem

Let $u \in C^k(\mathbb{T})$, $\int_{\mathbb{T}} u = 0$, $\alpha \in \mathbb{R}$, $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.
Find $v : \mathbb{T} \to \mathbb{R}$ such that

$$v(x+\alpha) - v(x) = u(x), \text{ for all } x \in \mathbb{T} \tag{1}$$

Classical Small Divisor Problem

In Fourier coefficients, (1) becomes

$$(e^{2\pi i n\alpha} - 1)\hat{v}(n) = \hat{u}(n), \quad n \in \mathbb{Z} \setminus \{0\}$$
$$\hat{v}(n) = (e^{2\pi i n\alpha} - 1)^{-1}\hat{u}(n) \approx (\{n\alpha\})^{-1}\hat{u}(n)$$

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First Problem: α may be rational.

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First Problem: α may be rational.

Second Problem: For any irrational α there are ∞-many rational $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Q: Can we get a lower bound for $|\alpha - \frac{p}{q}|$?

Classical Small Divisor Problem

Definition

$\alpha \notin \mathbb{Q}$ is a Diophantine number if $\exists c > 0$ and $r \geq 2$ such that

$$\left| \alpha - \frac{p}{q} \right| > cq^{-r}$$

for any $p/q \in \mathbb{Q}$, $q > 0$.

Note: Diophantine numbers have full measure.

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Note: Diophantine numbers have full measure.

Assuming α is Diophantine,

$$|\hat{v}(n)| \lesssim_c |n|^{r-1}|\hat{u}(n)|, \quad n \in \mathbb{Z} \setminus \{0\}$$

Consequence: Loss of regularity; $u \in H^k(\mathbb{T}) \Rightarrow v \in H^{k-r+1}(\mathbb{T})$.

Baby KAM

Consider the system

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} i\omega_1 & & \\ & \ddots & \\ & & i\omega_n \end{pmatrix}, \qquad \omega_j \in \mathbb{R}$$

If $\omega = (\omega_1, ..., \omega_n)$ is rationally independent, solutions given by $x_j(t) = c_j e^{i\omega_j t}$ are quasi-periodic. Does the periodic solution persist under perturbation?

Baby KAM

Perturb the system:

$$\dot{y} = Ay + g(y)$$

where

$$g(y) = \sum_{|k|_1 \geq 2} g_k y^k, \quad k \in \mathbb{N}^n \setminus \{0\}$$

where $|k|_1 = k_1 + \cdots + k_n$.

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Ansatz for periodic solution:

$$y(t) = u(e^{At}c), \qquad u(x) = x + \sum_{|k|_1 \geq 2} u_k x^k$$

Inserting the ansatz into the perturbed equation, we obtain

$$\sum_{|k|_1 \geq 2} (\omega \cdot k - A) u_k x^k = g\left(x + \sum_{|k|_1 \geq 2} u_k x^k\right)$$

Baby KAM

In coefficients, we have

$$(i\omega \cdot k - A)u_k = \sum_{\substack{|m|_1, |\ell|_1 \geq 2 \\ k_i = m_i \ell_i}} g_m u_\ell.$$

So we return to the same problem which can be resolved by imposing a similar Diophantine condition:

$$|\omega \cdot k - \omega_j| \geq \frac{\delta}{|k|_1^\tau} \tag{2}$$

for some $\delta, \tau > 0$. This set of frequency vectors ω satisfying (2) has full measure in $\mathbb{R}^n$.

Statement of a Problem

- Nonlinear Schrödinger Equation

$$i\partial_t u = \Delta u + \lambda |u|^{2p} u \tag{3}$$
$$x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}$$

Statement of a Problem

- Nonlinear Schrödinger Equation

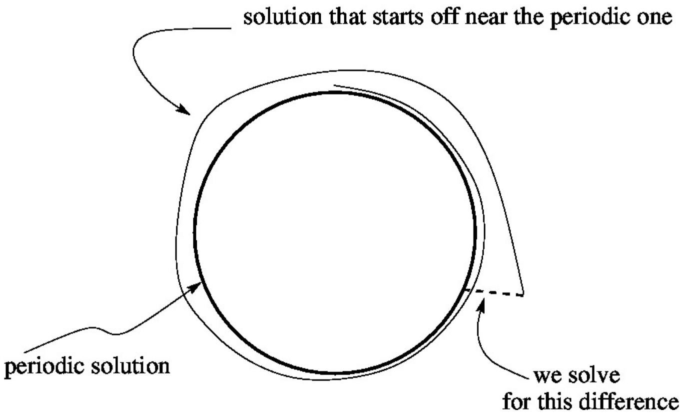
$$i\partial_t u = \Delta u + \lambda |u|^{2p} u \tag{3}$$
$$x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}$$

- Consider the plane wave solution to (3):

$$w_m(x,0) := \varrho e^{im\cdot x}$$
$$w_m(x,t) = \varrho e^{im\cdot x} e^{i(|m|^2 - \lambda \varrho^{2p})t}$$

- Assuming $u(x,t)$ satisfies (3) and $\|\varrho - e^{-im\cdot x} u(x,0)\|_{H^s(\mathbb{T}^d)} < \varepsilon$, what type of stability can we expect?

Picture


solution that starts off near the periodic one
periodic solution
we solve
for this difference

Our Goal

Definition (Orbital Stability)

A solution $x(t)$ is said to be orbitally stable if, given $\varepsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any other solution, $y(t)$, satisfying $|x(t_0) - y(t_0)| < \delta$, then $d(y(t), O(x_0, t_0)) < \epsilon$ for $t > t_0$.

- For any $M \in \mathbb{N}$
- There exist s_0 and ε_0 so that for any solution u to (3) with $\|\varrho - e^{-im\cdot x}u(x,0)\|_{H^s(\mathbb{T}^d)} < \varepsilon$, for $\varepsilon < \varepsilon_0$ and $s > s_0$
- $$\inf_{\varphi \in \mathbb{R}} \|e^{-i\varphi} e^{-im\cdot \bullet} w_m(\bullet, t) - e^{-im\cdot \bullet} u(\bullet, t)\|_{H^s(\mathbb{T}^d)} < \varepsilon C(M, s_0, \varepsilon_0)$$
- For $t < \varepsilon^{-M}$.

First Approach

- Assume $m = 0$
- Translation of (3) by w_0:

$$i\partial_t u = (\Delta + (p+1)\lambda\varrho^{2p})u + (p\lambda\varrho^{2(p-1)})w_0^2\bar{u} + \sum_{i=2}^{2p+1} F_i(u, \bar{u}, w_0) \tag{4}$$

$$i\partial_t u_n = (-|n|^2 + (p+1)\lambda\varrho^{2p})u_n + (p\lambda\varrho^{2(p-1)})w_0^2\bar{u}_{-n} + F(u_k, \bar{u}_k, w_0) \tag{5}$$

- The linear part of (5) is a system with periodic coefficients, so we consider Floquet's theorem.

Theorem (Floquet's Theorem)

Suppose $A(t)$ is periodic. Then the Fundamental matrix of the linear system has the form

$$\Pi(t, t_0) = P(t, t_0)\exp((t - t_0)Q(t_0))$$

where $P(\cdot, t_0)$ has the same period as $A(\cdot)$ and $P(t_0, t_0) = \mathbb{1}$.

The eigenvalues of $M(t_0) := \Pi(t_0 + T, t_0)$, ρ_j, are known as Floquet multipliers and

Corollary

A periodic linear system is stable if all Floquet multipliers satisfy $|\rho_j| \leq 1$.

Constant coefficients and Diagonalization

With $z_n = e^{-i\lambda\varrho^{2p}t}u_n$, the linear part of (5) is

$$i\partial_t \begin{pmatrix} z_n \\ \bar{z}_{-n} \end{pmatrix} = A_n \begin{pmatrix} z_n \\ \bar{z}_{-n} \end{pmatrix}$$

We then diagonalize

$$i\partial_t \begin{pmatrix} x_n \\ \bar{x}_{-n} \end{pmatrix} = \begin{pmatrix} \Omega_n & 0 \\ 0 & \Omega_{-n} \end{pmatrix} \begin{pmatrix} x_n \\ \bar{x}_{-n} \end{pmatrix}$$

where

$$\Omega_n = \sqrt{|n|^2(|n|^2 + 2p\varrho^{2p})}$$

assuming $\lambda = -1$.

Duhamel Iteration Scheme

Duhamel's Formula:

$$x_n(t) = e^{i\Omega_n t} x_n(0) + \int_0^t e^{i\Omega_n(t-s)} F(x(s))_n \, ds$$

Define the iteration scheme:

$$\begin{cases} x_n(t, k+1) = x_n(t, 0) + \int_0^t e^{i\Omega_n(t-s)} F(x_n(s, k)) \, ds \\ x_n(t, 0) := e^{i\Omega_n t} x_n(0, 0) \end{cases}$$

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- This approach is similar to the 19th century approach of expanding the solution in a perturbative series:

  $$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots$$

  u_k being defined recursively.
- This series does not converge, so we should expect a similar phenomenon.

Appearance of small divisors

The first step demonstrates issues that this iteration scheme presents us:

Small Model of First Iterate

$$x_n(t,1) = x_n(t,0) + \int_0^t e^{i\Omega_n(t-s)} \sum_{n_1,n_2} x_{n_1}(s,0)x_{n_2}(s,0)\, ds$$

$$= x_n(t,0) + e^{i\Omega_n t} \sum_{n_1,n_2} x_{n_1}x_{n_2} \int_0^t e^{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)s}\, ds$$

$$= x_n(t,0) + \sum_{n_1,n_2} x_{n_1}x_{n_2} \frac{e^{i(\Omega_{n_1}+\Omega_{n_2})t} - e^{i\Omega_n t}}{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)}$$

Recall that

$$\Omega_n = \sqrt{|n|^2(|n|^2 + 2p\varrho^{2p})}$$

and note the pattern

$$\partial_\varrho \Omega_n = \frac{C(n,\varrho)}{\sqrt{|n|^2 + 2p\varrho^{2p}}} = \Omega_n \frac{\tilde{C}(n,\varrho)}{|n|^2 + 2p\varrho^{2p}}$$

$$\partial_\varrho^2 \Omega_n = \frac{-C^2(n,\varrho)}{(|n|^2 + 2p\varrho^{2p})^{3/2}} = \Omega_n \frac{-\tilde{C}^2(n,\varrho)}{(|n|^2 + 2p\varrho^{2p})^2}$$

We can conclude that

$$\Omega_{n_1} + \Omega_{n_2} - \Omega_n = \partial_\varrho(\Omega_{n_1} + \Omega_{n_2} - \Omega_n) = \partial_\varrho^2(\Omega_{n_1} + \Omega_{n_2} - \Omega_n) = 0$$

does not occur when ϱ is restricted to a compact set.

Appearance of small divisors

Small Model of Second Iterate

$$
\begin{aligned}
&x_n(t,2)\\
&= x_n(t,0) + \int_0^t e^{i\Omega_n(t-s)} \sum_{n_1,n_2} x_{n_1}(s,1) x_{n_2}(s,1)\, ds\\
&= x_n(t,1)\\
&+ e^{i\Omega_n t} \sum_{n_1,k_1,k_2} \frac{x_{n_1} x_{k_1} x_{k_2} \int_0^t e^{i(\Omega_{n_1}+\Omega_{k_1}+\Omega_{k_2}-\Omega_n)s} - e^{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)s}\, ds}{i(\Omega_{k_1}+\Omega_{k_2}-\Omega_{n_2})}\\
&+ e^{i\Omega_n t} \sum_{j_1,j_2,k_1,k_2} \frac{x_{j_1} x_{j_2} x_{k_1} x_{k_2} \int_0^t e^{i(\Omega_{j_1}+\Omega_{j_2}+\Omega_{k_1}+\Omega_{k_2}-\Omega_n)s} - \ldots\, ds}{-(\Omega_{j_1}+\Omega_{j_2}-\Omega_{n_1})(\Omega_{k_1}+\Omega_{k_2}-\Omega_{n_2})}\\
&+ \ldots
\end{aligned}
$$

Issues

- Convergence
- Controlling loss of regularity
- Resonances
- Type of stability
    - Problem at zero mode

A Reduction on the Hamiltonian

$$H := \sum_{k\in\mathbb{Z}^d} |k|^2|u_k|^2 + \frac{1}{p+1} \sum_{\sum_{i=1}^{p+1} k_i = \sum_{i=1}^{p+1} h_i} u_{k_1}\dots u_{k_{p+1}} \bar{u}_{h_1}\dots\bar{u}_{h_{p+1}}. \tag{6}$$

Let $L := \|u(0)\|_{L^2}^2$, define the symplectic reduction of u_0:

$$\{u_k, \bar{u}_k\}_{k\in\mathbb{Z}^d} \to (L, \nu_0, \{v_k, \bar{v}_k\}_{k\in\mathbb{Z}^d\setminus\{0\}}),$$

$$u_0 = e^{i\nu_0}\sqrt{L - \sum_{k\in\mathbb{Z}^d} |v_k|^2}, \quad u_k = v_k e^{i\nu_0}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.$$

We now diagonalize the quadratic part of the Hamiltonian:

$$H_0 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} (k^2 + L^p p)|v_k|^2 + L^p \frac{p}{2}(v_k v_{-k} + \bar{v}_k \bar{v}_{-k}) \qquad (7)$$

which gives

$$H_0 = \sum_{k \in \mathbb{Z}^d} \frac{\Omega_k}{2}(|x_k|^2 + |x_{-k}|^2) \qquad (8)$$

with $\Omega_k = \sqrt{|k|^2(|k|^2 + 2pL^p)}$.

- It is convenient to group together the modes having the same frequency, i.e. to denote

$$\omega_q := \sqrt{q^2(q^2 + 2pL^p)}, \qquad q \geq 1. \qquad (9)$$

Birkhoff Normal Form Theorem in Finite Dimension

Definition (Normal Form)

Let $H = H_0 + P$ where $P \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$, which is at least cubic such that P is a perturbation of H_0. We say that P is in **normal form** with respect to H_0 if it Poisson commutes with H_0:

$$\{P, H_0\} = 0$$

Definition (Nonresonance)

Let $r \in \mathbb{N}$. A frequency vector, $\omega \in \mathbb{R}^n$, is **nonresonant up to order r** if

$$k \cdot \omega := \sum_{j=1}^{n} k_j \omega_j \neq 0 \text{ for all } k \in \mathbb{Z}^n \text{ with } 0 < |k| \leq r$$

Birkhoff Normal Form Theorem in Finite Dimension

Theorem (Moser '68)

Let $H = H_0 + P$ *where*

- $H_0 = \sum_{j=1}^{N} \omega_j \frac{p_j^2+q_j^2}{2}$
- $P \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ *having a zero of order 3 at the origin*

Fix $M \geq 3$ *an integer. There exists* $\tau : \mathcal{U} \ni (q', p') \mapsto (q, p) \in \mathcal{V}$ *a real analytic canonical transformation from a nbhd of the origin to a nbhd of the origin which puts* H *in normal form up to order* M *i.e.*

$$H \circ \tau = H_0 + Z + R$$

with

1. Z *is a polynomial of order* r *and is in normal form*
2. $R \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ *and* $R(z, \bar{z}) = O(\|(q, p)\|^{M+1})$
3. τ *is close to the identity:* $\tau(q, p) = (q, p) + O(\|(q, p)\|^2)$

Corollary

Assume ω is nonresonant. For each $M \geq 3$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\|(q_0, p_0)\| = \varepsilon < \varepsilon_0$ the solution $(q(t), p(t))$ of the Hamiltonian system associated to H which takes value (q_0, p_0) at $t = 0$ satisfies

$$\|(q(t), p(t)\| \leq 2\varepsilon \text{ for } |t| \leq \frac{c}{\varepsilon^{M-1}}.$$

Normal Form: Formal Argument

Consider the ODE

$$i\partial_t x_n = \omega_n x_n + \sum_{k\geq 2} \left(f_k(x)\right)_n$$

With

- Auxiliary Hamiltonian: $\chi(x)$
- X_χ the corresponding vector field

We note that for any vector field Y, its transformed vector field under the time 1 flow generated by X_χ is

$$e^{\mathrm{ad}_{X_\chi}} Y = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}_{X_\chi}^k Y \qquad (10)$$

where $\mathrm{ad}_X Y := [Y, X]$.

Iterative Step

- Let χ be degree $K_0 + 1$
- Let $\Phi_\chi(x)$ be the time-1 flow map associated with the Hamiltonian vector field X_χ.
- Consider the change of variables $y = \Phi_\chi(x)$
- Using the identity (10), one obtains

$$i\partial_t y_n = \omega_n y_n + \sum_{k=2}^{K_0-1} \left(f_k(y)\right)_n + ([X_\chi, \omega y](y))_n + (f_{K_0}(y))_n + h.o.t.$$

Plan: choose χ and another vector-valued homogeneous polynomial of degree K_0, R_{K_0}, in such a way that we can decompose f_{K_0} as follows

$$f_{K_0}(y) = R_{K_0}(y) - [X_\chi, \omega y](y) \tag{11}$$

- We can find χ so that R_{K_0} is in the kernel of the following function

$$\mathrm{ad}_\omega(X) := [X, \omega y].$$

- Any $Y \in \ker \mathrm{ad}_\omega$ is referred to as "normal" or "resonant".

Appearance of small divisors

- Condition for a monomial, $y^\alpha \bar{y}^\beta \partial_{y_m}$, $(\alpha, \beta \in \mathbb{N}^\infty)$ to satisfy $y^\alpha \bar{y}^\beta \partial_{y_m} \in \ker \mathrm{ad}_\omega$:

$$\mathrm{ad}_\omega(y^\alpha \bar{y}^\beta \partial_{y_m}) = [(\alpha - \beta) \cdot \omega - \omega_m] y^\alpha \bar{y}^\beta \partial_{y_m}$$

- For individual terms, (11) becomes

$$R_{\alpha,\beta,m} - (\omega \cdot (\alpha - \beta) - \omega_m) X_{\alpha,\beta,m} = f_{\alpha,\beta,m}$$

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- Definition of X_χ and R_{K_0}:

$$\begin{matrix} R_{\alpha,\beta,m} := f_{\alpha,\beta,m} \\ X_{\alpha,\beta,m} := 0 \end{matrix} \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m = 0$$

$$X_{\alpha,\beta,m} := \frac{-f_{\alpha,\beta,m}}{(\omega \cdot (\alpha - \beta) - \omega_m)} \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m \neq 0$$

Appearance of small divisors

- In finite dimension,

$$\inf\{|\omega\cdot(\alpha-\beta)-\omega_m| \,\big|\, \omega\cdot(\alpha-\beta)-\omega_m\neq 0\}>0$$

- Leads to bound on change-of-variables map (symplectomorphism).
- Not necessarily true in infinite dimensions.

Nonresonance Condition

Definition (Nonresonance Condition)

There exists $\gamma = \gamma_M > 0$ and $\tau = \tau_M > 0$ such that for any N large enough, one has

$$\left| \sum_{q \geq 1} \lambda_q \omega_q \right| \geq \frac{\gamma}{N^\tau} \qquad \text{for } \|\lambda\|_1 \leq M, \ \sum_{q>N} |\lambda_q| \leq 2 \tag{12}$$

where $\lambda \in \mathbb{Z}^\infty \setminus \{0\}$.

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where $\lambda \in \mathbb{Z}^\infty \setminus \{0\}$.

The following generalization of the "non-resonance" result in Bambusi-Grebert holds.

Theorem (Bambusi-Grebert 2006)

For any $L_0 > 0$, there exists a set $J \subset (0, L_0)$ of full measure such that if $L \in J$ then for any $M > 0$ the Nonresonance Condition holds.

Ideas behind Nonresonance

Recall that $\omega_q = \sqrt{q^2(q^2 + 2p\rho^{2p})}$. For any $K \le N$, consider K indices $j_1 < \cdots < j_K \le N$. Then

$$\begin{vmatrix} \omega_{j_1} & \omega_{j_2} & \cdots & \omega_{j_K} \\ \frac{d}{dm}\omega_{j_1} & \frac{d}{dm}\omega_{j_2} & \cdots & \frac{d}{dm}\omega_{j_K} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{d^{K-1}}{dm^{K-1}}\omega_{j_1} & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_2} & \cdots & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_K} \end{vmatrix} \gtrsim N^{-2K^2}$$

where $m = \rho^{2p}$.

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$$\left| \begin{matrix} \omega_{j_1} & \omega_{j_2} & \cdots & \omega_{j_K} \\ \frac{d}{dm}\omega_{j_1} & \frac{d}{dm}\omega_{j_2} & \cdots & \frac{d}{dm}\omega_{j_K} \\ . & . & \cdots & . \\ \frac{d^{K-1}}{dm^{K-1}}\omega_{j_1} & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_2} & \cdots & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_K} \end{matrix} \right| \gtrsim N^{-2K^2}$$

where $m = \rho^{2p}$.

Consequently, for $\alpha > 50r^3$. $\forall \gamma > 0$ small enough, $\exists J_\gamma \subset [m_0, m_1]$ such that for all $m \in J_\gamma$, for all $N \geq 1$,

$$\left| \sum_{j=1}^{N} k_j \omega_j + n \right| \geq \frac{\gamma}{N^\alpha}$$

$\forall k \in \mathbb{Z}^N$ with $0 \neq |k| \leq r$ and $\forall n \in \mathbb{Z}$. Moreover,

$$|[m_0, m_1] \setminus J_\gamma| \lesssim \gamma^{1/r}$$

Definition

For $x = \{x_n\}_{n \in \mathbb{Z}^d}$, define the standard Sobolev norm as

$$\|x\|_s := \sqrt{\sum_{n \in \mathbb{Z}^d} |x_n|^2 \langle n \rangle^{2s}}$$

Define H^s as

$$H^s := \{x = \{x_n\}_{n \in \mathbb{Z}^d} \mid \|x\|_s < \infty\}$$

Normal Form Theorem

Theorem (Bambusi-Grebert 2006)

Consider the equation

$$i\dot{x} = \omega x + \sum_{k \geq 2} f_k(x). \tag{13}$$

and assume the nonresonance condition (12). *For any* $M \in \mathbb{N}$, *there exists* $s_0 = s_0(M, \tau)$ *such that for any* $s \geq s_0$ *there exists* $r_s > 0$ *such that for* $r < r_s$, *there exists an analytic canonical change of variables*

$$y = \Phi^{(M)}(x)$$
$$\Phi^{(M)} : B_s(r) \to B_s(3r)$$

which puts (13) *into the normal form*

$$i\dot{y} = \omega y + \mathcal{R}^{(M)}(y) + \mathcal{X}^{(M)}(y). \tag{14}$$

Normal Form Theorem continued

Theorem (Theorem cont.)

Moreover there exists a constant $C = C_s$ such that:

- $$\sup_{x \in B_s(r)} \|x - \Phi^{(M)}(x)\|_s \le Cr^2$$
- $\mathcal{R}^{(M)}$ *is at most of degree $M+2$, is resonant, and has tame modulus*
- *the following bound holds*

$$\|\mathcal{X}^{(M)}\|_{s,r} \le Cr^{M+\frac{3}{2}}$$

Main Theorem: Statement from FGL '13

Theorem (Faou, Gauckler, Lubich 2013)

Let $\rho_0 > 0$ be such that $1 - 2\lambda\rho_0^2 > 0$, and let $M > 1$ be fixed arbitrarily. There exists $s_0 > 0$, $C \geq 1$ and a set of full measure $\mathcal{P}$ in the interval $(0, \rho_0]$ such that for every $s \geq s_0$ and every $\rho \in \mathcal{P}$, there exists ε_0 such that for every $m \in \mathbb{Z}^d$ the following holds: if the initial data $u(\bullet, 0)$ are such that

$$\|u(\bullet, 0)\|_{L^2} = \rho \qquad \text{and} \qquad \|e^{-im\cdot\bullet}u(\bullet, 0) - u_m(0)\|_{H^s} = \varepsilon \leq \varepsilon_0$$

then the solution of (3) (with $p = 1$) with these initial data satisfies

$$\|e^{-im\cdot\bullet}u(\bullet, t) - u_m(t)\|_{H^s} \leq C\varepsilon \text{ for } t \leq \varepsilon^{-M}$$

Structure of the cubic case

Let

$$H_c = \int_{\mathbb{T}} (|\partial_x u|^2 + |u|^4)\, dx$$

Theorem (Kappeler, Grebert 2014)

There exists a bi-analytic diffeomorphism $\Omega : H^1 \to H^1$ such that Ω introduces Birkhoff coordinates for NLS on H^1. That is, on H^1 the transformed NLS Hamiltonian $H_c \circ \Omega^{-1}$ is a real-analytic function of the actions

$$I_n = \frac{|x_n|^2}{2}$$

for $n \in \mathbb{Z}$. Furthermore, $d_0\Omega$ is the Fourier transform.

Main Theorem: Statement

Theorem (W. 2014)

Let $L_0 > 0$ be such that $1 - 2p\lambda L_0^p > 0$, and let $M > 1$ be fixed arbitrarily. There exists $s_0 > 0$, $C \geq 1$ and a set of full measure $\mathcal{P}$ in the interval $(0, L_0]$ such that for every $s \geq s_0$ and every $L \in \mathcal{P}$, there exists ε_0 such that for every $m \in \mathbb{Z}^d$ the following holds: if the initial data $u(\bullet, 0)$ are such that

$$\|u(\bullet,0)\|_{L^2}^2 = L \quad \text{and} \quad \|e^{-im\cdot\bullet}u(\bullet,0) - u_m(0)\|_{H^s} = \varepsilon \leq \varepsilon_0$$

then the solution of (3) with these initial data satisfies

$$\|e^{-im\cdot\bullet}u(\bullet,t) - u_m(t)\|_{H^s} \leq C\varepsilon \text{ for } t \leq \varepsilon^{-M}$$

Proposition

The truncation of (14),

$$i\dot{y} = \omega y + \mathcal{R}^{(M)}(y)$$

can be decoupled in the following way:

$$i\partial_t \begin{pmatrix} y_{n_1} \\ \cdots \\ y_{n_k} \end{pmatrix} = \mathcal{M}_q \begin{pmatrix} y_{n_1} \\ \cdots \\ y_{n_k} \end{pmatrix} \qquad (15)$$

where $q \geq 1$, $\{n_1, \dots, n_k\} := \{n \in \mathbb{Z}^d \,:\, |n| = q\}$, $\mathcal{M}_q = \mathcal{M}_q\,(\omega, \{y_j\})$ *is a self-adjoint matrix for all* t.

Further Questions

- Infinite time result?
- Feasibility of the Floquet/Duhamel iteration
- KAM result

KAM Result

- Obstacles
  - One parameter family of frequencies
  - Repeated frequencies
- May be able to overcome this: Bambusi, Berti, Magistrelli *Degenerate KAM theory for PDEs*

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Thank you for listening