Finally, we have enough to state a thm:

<u>**Thm:**</u> (Bendt, Graf and Schief)

Let Φ be a self-similar IFS and let $s = \text{sim-dim}(\Phi)$. Then TFAE:

i.) Condition BG1

ii.) Condition BG2

iii.) Condition BG3

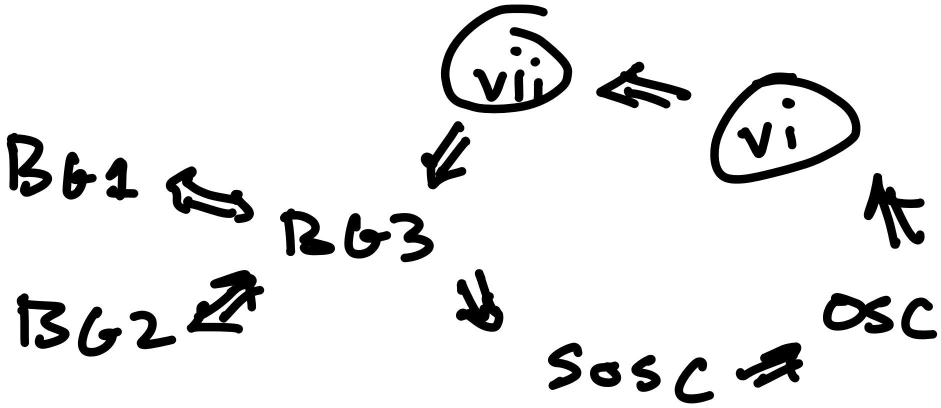
iv) SOSC

v.) OSC

vi.) $0 < \mathcal{H}^s(\Lambda)$

vii.) Λ is s-Ahlfors regular.

Plan



First, we prove

$$BG3 \Rightarrow SOSC \Rightarrow OSC \Rightarrow (vi) \Rightarrow (vii) \Rightarrow BG3.$$

1) BG3 $\Rightarrow$ SOSC.

pf: Let $\varepsilon > 0$ such that $(1+2\varepsilon) r_{max} < 1$

and

$$\sup_{\underline{k} \in \Sigma^*} \# \tilde{\Pi}_\varepsilon(\underline{k}) < \infty$$

Let $\underline{k}_0 \in \Sigma^*$ be such that

$$\# \tilde{\Pi}_\varepsilon(\underline{k}_0) = \max_{\underline{k} \in \Sigma^*} \# \tilde{\Pi}_\varepsilon(\underline{k}).$$

Note that $\tilde{\Pi}_\varepsilon$ has the following stability condition:

$$\tilde{\Pi}_\varepsilon(\underline{i}\,\underline{j}) \supset \{ \underline{i}\,\underline{k} \mid \underline{k} \in \tilde{\Pi}_\varepsilon(\underline{j}) \}.$$
$$= \underline{i} \cdot \tilde{\Pi}_\varepsilon(\underline{j}).$$

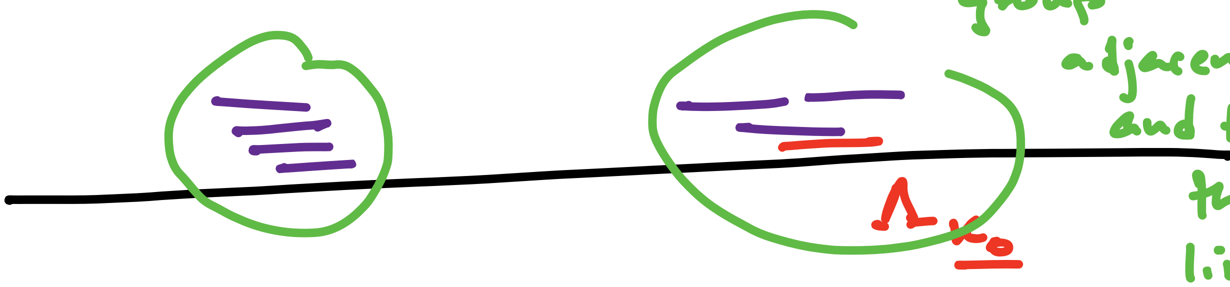
The maximality of $\underline{k}_0$ implies for any $\underline{i} \in \Sigma^*$

$$\tilde{\Pi}_\varepsilon(\underline{i}\,\underline{k}_0) = \underline{i} \cdot \tilde{\Pi}_\varepsilon(\underline{k}_0).$$

Thus any j s.t. $j \in \check{\Gamma}_\varepsilon(\underline{i}\,k_0)$

$$\left(\begin{array}{l} \operatorname{diam}(\Lambda_j) \sim \phi_{\underline{i}k_0}(N_\varepsilon(\Lambda)) \\ \text{and} \qquad \Lambda_j \cap \phi_{\underline{i}k_0}(N_\varepsilon(\Lambda)) \neq \emptyset \end{array}\right).$$

j must have $\underline{i}$ as a prefix.



group adjacent pieces and treat them like the IFS.

$\Rightarrow$ If $j_1 \neq i_1$ then

$$\operatorname{dist}(\Lambda_{j_1}, \Lambda_{\underline{i}k_0}) \geq \varepsilon\, r_{\underline{i}k_0}$$

Now

$$V := \bigcup_{\underline{i} \in \Sigma^*} \phi_{\underline{i}k_0}(N_{\varepsilon/2}(\Lambda))$$

satisfies the SOSC $\square$

2) SOSC $\Rightarrow$ OSC

Obvious

3.) OSC $\Rightarrow$ $\mathcal{H}^s(\Lambda) > 0$

Let V be a nonempty open set satisfying OSC.

then

$$E_{N+1} := \bigcup_{i=1}^{m} \phi_i(E_N)$$

$$E_0 := \overline{V}.$$

Then

$$\bigcap_{N=1}^{\infty} E_{N+1} = \Lambda$$

The proof is very similar to the SSP $\Rightarrow$ $\mathcal{H}^s(\Lambda) > 0$ proof, so we skip it.

4.) $\mathcal{H}^s(\Lambda) > 0$ $\Rightarrow$ Λ is Ahlfors regular

$$\mathcal{H}^s(\Lambda) > 0 \quad \Rightarrow \quad s \leq \dim_{\mathcal{H}}(\Lambda)$$

Since $\dim_{\mathcal{H}}(\Lambda) \leq \text{sim-dim}(\Phi)$,

$$s = \dim_{\mathcal{H}}(\Lambda).$$

First, if $\mathcal{H}^s_\infty(E) = \inf\left\{ \sum \operatorname{diam}(E_i)^s \,\middle|\, E \subset \bigcup_i E_i \right\}$

we call $\mathcal{H}^s_\infty(E)$ the s-dim Hausdorff content.

Then $\mathcal{H}^s_\infty$ is subadditive (Not additive).

and $\mathcal{H}^s_\infty(E) \leq \mathcal{H}^s(E)$.

If $\mathcal{H}^s(E)=0$ then $\mathcal{H}^s_\infty(E)=0$.

Observe that $\{\phi_{\underline{i}}(\Lambda)\}_{\underline{i} \in \Sigma_1^*}$ is a Vitali Covering. Vitali covering implies

$\exists\, I \subset \Sigma_1^{\prime *}$ such that $\phi_{\underline{i}}(\Lambda) \cap \phi_{\underline{j}}(\Lambda) = \emptyset$

for $\underline{i} \neq \underline{j} \in I$ and

either

- $\mathcal{H}^s\left(\Lambda \setminus \bigcup_{\underline{i} \in I} \phi_{\underline{i}}(\Lambda)\right) = 0$

or

- $\sum \operatorname{diam}(\phi_{\underline{i}}(\Lambda))^s$
$$= \operatorname{diam}(\Lambda)^s \sum_{\underline{i} \in I} r_{\underline{i}}^s = \infty$$

Note that $\mathcal{H}^s(\Lambda) < \infty$ (by the sim dimension calculation)

so

$$\mathcal{H}^s(\Lambda) \sum_{\underline{i} \in I} r_{\underline{i}}^s = \sum_{\underline{i} \in I} \mathcal{H}^s(\phi_{\underline{i}}(\Lambda))$$

$$= \mathcal{H}^s\left(\bigcup_{\underline{i} \in I} \phi_{\underline{i}}(\Lambda) \right)$$

$$\leq \mathcal{H}^s(\Lambda) < \infty .$$

$$\Rightarrow \quad \mathcal{H}^s\left(\Lambda \setminus \bigcup_{\underline{i} \in I} \phi_{\underline{i}}(\Lambda)\right) = 0 .$$

$$\Leftrightarrow \mathcal{H}^s(\Lambda) = \sum_{\underline{i} \in I} \mathcal{H}^s(\phi_{\underline{i}}(\Lambda))$$

$$= \mathcal{H}^s(\Lambda) \sum_{\underline{i} \in I} r_{\underline{i}}^s$$

$$\Rightarrow \quad \sum r_{\underline{i}}^s = 1 .$$

Let $\varepsilon > 0$, $\delta > 0$ and let $\{U_\ell\}_{\ell=1}^{\infty}$ be a cover of Λ s.t.

$$\sum \operatorname{diam}(U_\ell)^s \leq \mathcal{H}^s_\infty(\Lambda) + \varepsilon .$$

Let n be such that

$$\max_\ell \operatorname{diam}(U_\ell) < \delta \cdot \left(\max_{\underline{i}} r_{\underline{i}} \right)^{-n}$$

Then , since

$$\mathcal{H}^s(\Lambda) = \mathcal{H}^s\Big(\bigcup_{\underline{i}_1,\dots,\underline{i}_n \in I} \phi_{\underline{i}_1}\circ\cdots\circ\phi_{\underline{i}_n}(\Lambda)\Big)$$

$$\mathcal{H}^s_\delta(\Lambda) \le \mathcal{H}^s_\delta\Big(\bigcap_n \bigcup \phi_{\underline{i}_1}\circ\cdots\circ\phi_{\underline{i}_n}(\Lambda)\Big)$$

$$\le \sum_{\underline{i}_1,\dots,\underline{i}_n \in I,\ell} \operatorname{diam}\big(\phi_{\underline{i}_1}\circ\cdots\circ\phi_{\underline{i}_n}(U_\ell)\big)^s$$

$$= \Big(\sum_{\underline{i}\in I} r_{\underline{i}}^s\Big)^n \sum \operatorname{diam}(U_i)^s$$

$$\le \mathcal{H}^s_\infty(\Lambda) + \varepsilon.$$

$$\Rightarrow \mathcal{H}^s(\Lambda) = \mathcal{H}^s_\infty(\Lambda)$$

For any $E \subset \Lambda$,

$$\mathcal{H}^s_\infty(E) \le \mathcal{H}^s(E) = \mathcal{H}^s(\Lambda) - \mathcal{H}^s(\Lambda \setminus E)$$

$$\le \mathcal{H}^s_\infty(\Lambda) - \mathcal{H}^s_\infty(\Lambda \setminus E)$$

$$\le \mathcal{H}^s_\infty(E)$$

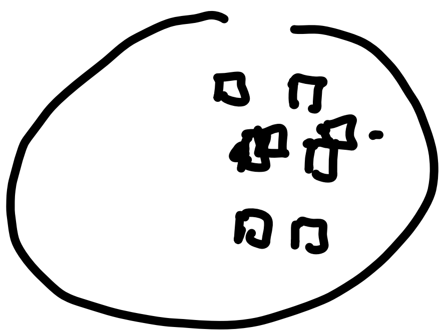
$\Rightarrow \mathcal{H}^s_\infty(E) = \mathcal{H}^s(E)$ for $E \subset \Lambda$

Therefor, for $x \in \Lambda$ $r \in (0,\infty)$

$$\mathcal{H}^s(E \cap B(x,r))$$
$$\le \mathcal{H}^s_\infty(E \cap B(x,r))$$
$$\le \mathcal{H}^s_\infty(B(x,r)) \lesssim r^s .$$

For the lower bound, let $x \in \Lambda$

$r > 0$.



For N large enough, $\exists\, \underline{i} \in \Sigma^N$ s.t.

$r_{\underline{i}} \sim r$, $\exists\, \underline{j} \in \Sigma^{\mathbb{N}}$ s.t.
$\Pi(\underline{i}\,\underline{j}) = x$

$$\Rightarrow \quad B(x,r) \supset \Lambda_{\underline{i}}$$

$$\Rightarrow \quad \mathcal{H}^s(B(x,r) \cap \Lambda) \ge \mathcal{H}^s(\Lambda_{\underline{i}})$$
$$= r_{\underline{i}}^s \, \mathcal{H}^s(\Lambda)$$
$$\gtrsim r^s \cdot \mathcal{H}^s(\Lambda). \quad \square$$

5.) Ahlfors regularity $\Rightarrow$ BG3.

Pf: Λ Ahlfors regular implies $\exists C \geq 1$ s.t.

$$C^{-1} r^s \leq \mathcal{H}^s(\Lambda \cap B(x,r)) \leq C r^s$$

$\forall r \in [0, \mathrm{diam}(\Lambda))$ and $\forall x \in \Lambda$

Now

$$0 < \mathcal{H}^s(\Lambda) = \mathcal{H}^s\left(\bigcup \phi_i(\Lambda)\right)$$
$$\leq \sum \mathcal{H}^s(\phi_i(\Lambda))$$
$$= \sum r_i^s \mathcal{H}^s(\Lambda) = \mathcal{H}^s(\Lambda) < \infty$$

$$\Rightarrow \mathcal{H}^s\left(\bigcup \phi_i(\Lambda)\right) = \sum \mathcal{H}^s(\phi_i(\Lambda)).$$

$$\Rightarrow \mathcal{H}^s(\phi_i(\Lambda) \cap \phi_j(\Lambda)) = 0 \quad \text{for } i \neq j$$

Fix $\underline{i} \in \Sigma^*$, let $x \in \Lambda_{\underline{i}}$

For any $\underline{j} \in \hat{\Gamma}_\varepsilon(\underline{i})$, we have

$$\mathrm{diam}(\Lambda_{\underline{j}}) \leq \mathrm{diam}(N_{r_{\underline{i}}\varepsilon}(\Lambda_{\underline{i}}))$$

and

$$\Lambda_{\underline{j}} \cap N_{r_{\underline{i}}\varepsilon}(\Lambda_{\underline{i}}) \neq \emptyset$$

$$\Rightarrow \quad \Delta_j \subset B(x, 2\,\mathrm{diam}(N_{r_{\underline{i}}\varepsilon}(\Delta_{\underline{i}})))$$

Thus,

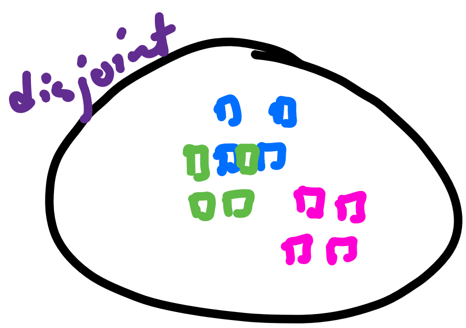
$$\mathrm{diam}(N_{r_{\underline{i}}\varepsilon}(\Delta_{\underline{i}}))^s$$

$$\gtrsim \mathcal{H}^s(\Delta, B(x, 2(\mathrm{diam}(N_{r_{\underline{i}}\varepsilon}(\Delta_{\underline{i}})))))$$

$$\geq \mathcal{H}^s\Big(\bigcup_{j \in \hat{\Gamma}_\varepsilon(\underline{i})} \Delta_j\Big)$$

$$= \sum_{j \in \hat{\Gamma}_\varepsilon(\underline{i})} r_j^s \, \mathcal{H}^s(\Delta)$$

$$\gtrsim \mathrm{diam}(\phi_{\underline{i}}(N_\varepsilon(\Delta))^s \; \#\hat{\Gamma}_\varepsilon(\underline{i}).$$



$$\Rightarrow \quad \#\hat{\Gamma}_\varepsilon(\underline{i}) \lesssim \frac{\mathrm{diam}(N_{r_{\underline{i}}\varepsilon}(\Delta_{\underline{i}}))^s}{\mathrm{diam}(\phi_{\underline{i}}(N_\delta(\Delta))^s} \lesssim 1 \qquad \square.$$