Thm: (Moran '46, Hutchinson '81)
Assume that the self-similar IFS,
$\Phi = (\phi_1, \dots, \phi_m)$ acts on $\mathbb{R}^d$ and satisfies the SSP. Then, if $s = \text{sim-dim}(\Phi)$,

$$0 < \mathcal{H}^s(\Lambda) < \infty \quad \text{and}$$

$$\dim_{\mathcal{H}}(\Lambda) = \dim_B(\Lambda) = s.$$

Pf:
We prove the statement in two steps:

1. $0 < \mathcal{H}^s(\Lambda) < \infty$ $(\Rightarrow \dim_{\mathcal{H}}(\Lambda) \geq s)$

2. $\overline{\dim}_B(\Lambda) \leq s$ (Now $s \leq \dim_{\mathcal{H}}(\Lambda) \leq \underline{\dim}_B \leq \overline{\dim}_B \leq s$).

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1. $$\mathcal{H}^s_\delta(\Lambda) \leq \limsup_{n\to\infty} \sum_{\underline{i} \in \Sigma_n} \operatorname{diam}(\Lambda_{\underline{i}})^s$$

$$\leq \limsup_{n\to\infty} \operatorname{diam}(\Lambda)^s \left(\sum_{j=1}^m r_j^s \right)^n$$

$$= \operatorname{diam}(\Lambda)^s < \infty \qquad \text{for all } \delta$$

$\Rightarrow \quad \mathcal{H}^s(\Lambda) < \infty.$

In order to prove $\mathcal{H}^s(\Lambda) > 0$, we define a measure on $\Sigma_1^+ = \{1, \dots, m\}^{\mathbb{N}}$.

Recall: $[i_1 \cdots i_n] = \{ \underline{j} \in \Sigma_1^+ \mid j_k = i_k \text{ for } k = 1, \dots, n \}.$

Define

$$\mu([i_1 \cdots i_n]) = r_{i_1}^s \cdots r_{i_n}^s$$

Then $\displaystyle\sum_{\underline{i} \in \Sigma_n} \mu([\underline{i}]) = (r_1^s + \dots + r_m^s)^n = 1.$

μ can be extended to a Borel measure on Σ_1^+.

Recall: Natural projection $\pi : \Sigma_1^+ \to \Lambda$

Since Φ satisfies SSP, π is a continuous bijection.

Define the push-forward measure

$$\nu(A) := \mu(\pi^{-1}(A))$$

Thus, for $\underline{i} \in \Sigma_n$

$$\nu(\Lambda_{\underline{i}}) = \mu([\underline{i}]) = r_{i_1}^s \cdots r_{i_n}^s$$

The SSP implies that

$$\ell := \min\{\operatorname{dist}(\Lambda_i, \Lambda_j) \mid i \neq j\} > 0.$$

Then, for $\underline{i} \in \Lambda_n$

$$\min\{\operatorname{dist}(\Lambda_{\underline{i}j}, \Lambda_{\underline{i}k}) \mid k \neq j\} = \ell r_{i_1} \cdots r_{i_n} \quad (*)$$

Now it suffices to show $\exists C > 0$ s.t.

$$\nu(B(x,r)) \leq C r^s$$

for all $x \in \Lambda$ and r small enough.

Fix $x \in \Lambda$, $r < \ell$, and let $\underline{i} = \pi^{-1}(x)$.

Let n be such that

$$\ell \, r_{i_1} \cdots r_{i_n} \leq r \leq r_{i_1} \cdots r_{i_{n-1}} \ell$$

$(*)$ implies

$$B(x,r) \cap \Lambda = B(x,r) \cap \Lambda_{i_1 \cdots i_{n-1}}.$$

Thus

$$\nu(B(x,r)) \le \nu(\Lambda_{i_1 \dots i_{n-1}}) \le r_{i_1}^s \cdots r_{i_{n-1}}^s \le \left(r_{min}^{-s}\right) r_{i_1}^s \cdots r_{i_{n-1}}^s \le (\rho\, r_{min})^{-s} \; r^s .$$

2

Lemma: Let Λ be the attractor of an IFS $\Phi = (\phi_1, \dots, \phi_m)$. Then

$$\overline{\dim}_B(\Lambda) \le \text{sim-dim}(\Phi).$$

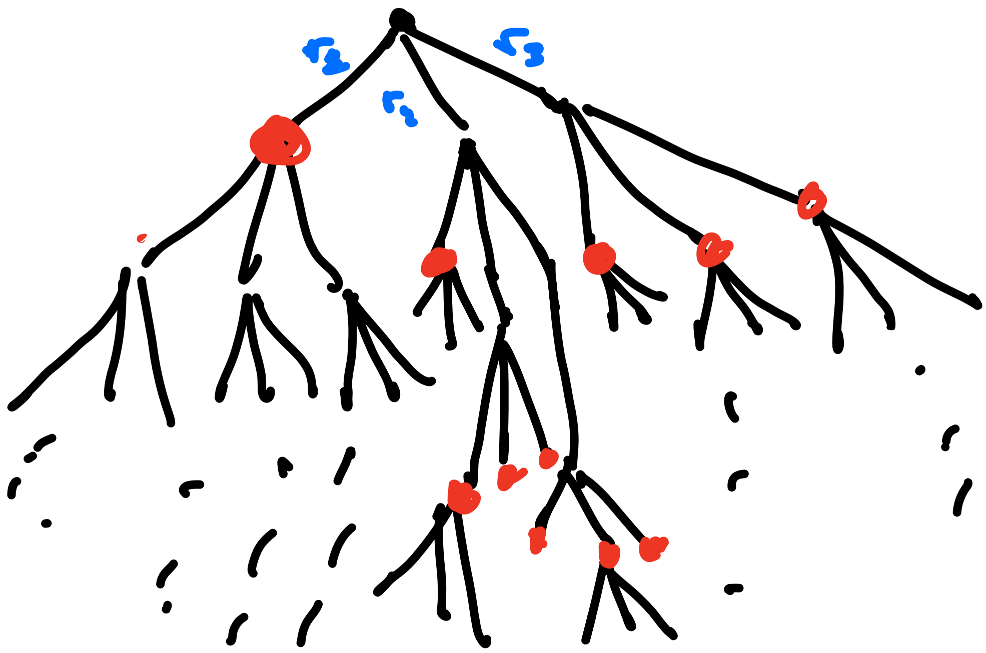
Proof of Lemma:

For $k \in \mathbb{Z}_+$, we define a Moran cut set Moran cut set of Σ'^*:

$$M_k := \{ \underline{i} \in \Sigma'^* \mid r_{i_1} \cdots r_{i_n} \le 2^{-k} < r_{i_1} \cdots r_{i_{n-1}} \}.$$

The associated cylinder set form a partition of Σ_1

<u>Example:</u> Let $n=3$, $0 < r_1 \ll r_2 < r_3 < 1$



I.e $\Sigma_1 = \bigcup_{\underline{i} \in M_k} [\underline{i}]$.

Now for all $\underline{i} \in M_k$, $\mathrm{diam}(\Lambda_{\underline{i}}) \sim 2^{-k}$ thus since $\Lambda_{\underline{i}_1} \cap \Lambda_{\underline{i}_2}$ for $\underline{i}_1, \underline{i}_2 \in M_k$ be the strong separation property,

$$\#\{q \in \Delta_k \mid q \cap \Lambda \neq \phi\} \sim \#M_k.$$

Claim:

Since $\{[\underline{i}] \mid \underline{i} \in M_k\}$ is a partition,

$$\sum_{\underline{i} \in M_k} r_{\underline{i}}^s = 1$$

Exercise.

Since $r_{\underline{i}} \sim 2^{-k}$,

$$\sum_{\underline{i} \in M_k} r_{\underline{i}}^s = 1 \implies (\# M_k) \cdot (2^{-k})^s \sim 1.$$

$$\implies \frac{\log\left(\# \{q \in \Delta_k \mid q \cap \Lambda \neq \emptyset\}\right)}{k \log 2} \leq \frac{\log(\# M_k)}{k \log(2)}$$

$$\leq \frac{s\, k \log 2}{k \log(2)} \leq s.$$

$\square$.

Similar Statement:

Thm: Let $E \subset \mathbb{R}^d$ if $\exists\ \mu$ supported in E s.t. $\mu(\mathbb{R}^d) \in (0, \infty)$ and $\exists\ C > 1$ s.t. $\forall x \in E$

$$C^{-1} r^s \leq \mu(B(x, r)) \leq C r^s$$

then $\dim_H(E) = s = \dim_P(E)$.

The open set condition and various overlapping conditions

The following is a discussion of a culmination of many results leading to multiple characterizations of the open set condition.

First, we establish some "overlap" conditions on an IFS.

The Bandt and Graf Conditions

Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be an IFS on $\mathbb{R}^d$.

I have forgotten to emphasize before that
$\|\phi_j(x) - \phi_j(y)\| = r_j\|x-y\|$

So, in particular, for each j, $\exists\, r_j \in (0,1)$, $A_j \in O(d)$ and $x_j \in \mathbb{R}^d$ such that

$$\phi_j(x) = r_j A_j x + x_j$$

Let $\Sigma_N := \{1, \dots, m\}^N$, $\Sigma_1 := \{1, \dots, m\}^{\mathbb{N}}$

and $\Sigma_1^* = \bigcup_{N \geq 1} \Sigma_N$

For $\underline{i} \in \Sigma_N$, $\phi_{\underline{i}} := \phi_{i_N} \circ \phi_{i_{N-1}} \circ \dots \phi_{i_1}$

<u>Condition BG1</u>

Consider the set of <u>neighbor maps</u>

$$\mathcal{N} := \{ \phi_{\underline{i}}^{-1} \circ \phi_{\underline{j}} \mid \underline{i} \neq \underline{j} \in \Sigma_1^* \}.$$

Thus $\mathcal{N} \subset \{$metric space of similitudes$\}$

Condition BG1 holds if

$$Id \notin clos(\mathcal{N})$$

<u>Condition BG2</u> For $\varepsilon > 0$ and $\underline{i}, \underline{j} \in \Sigma_1^*$, $\phi_{\underline{i}}$ and $\phi_{\underline{j}}$ are <u>ε-relatively close</u> if

$$\| \phi_{\underline{i}}(x) - \phi_{\underline{j}}(x) \| \leq \varepsilon \cdot \min(diam(\Lambda_{\underline{i}}), diam(\Lambda_{\underline{j}}))$$

for all $x \in \Lambda$.

Condition BG2 holds if $\exists\, \varepsilon > 0$ s.t. for each $\underline{i}, \underline{j} \in \Sigma_1^*$

$\phi_{\underline{i}}$ and $\phi_{\underline{j}}$ are <u>not</u> ε-relatively close

Condition BG3

Fix $\varepsilon \in (0,1)$ satisfying $(1+2\varepsilon)\, r_{max} < 1$

Define

$$\hat{\Gamma}_\varepsilon(\underline{i}) := \left\{ \underline{j} \in \Sigma_1^* \;\middle|\; \begin{array}{l} \underline{j} = (j_1, \dots, j_n) \\ \operatorname{diam}(\Delta_{\underline{j}}) \le \phi_{\underline{i}}(N_\varepsilon(\Delta)) < \operatorname{diam}(\Delta_{(j_1, \dots, j_{n-1})}) \\ \Delta_{\underline{j}} \cap \phi_{\underline{i}}(N_\varepsilon(\Delta)) \neq \emptyset \end{array} \right\}$$

Then define

$$\hat{\gamma}_\varepsilon := \sup_{\underline{i} \in \Sigma^*} \# \hat{\Gamma}_\varepsilon(\underline{i}).$$

Condition BG3 holds if $\hat{\gamma}_\varepsilon < \infty$ for some $\varepsilon \in \left(0, \min\left(1, \frac{r_{max}^{-1} - 1}{2}\right)\right)$

(we see that the condition on ε only is enforced if $r_{max} \approx 1$..)

A Note on the topology of the set of similitudes

For the set of functions, G, of the form $g: \mathbb{R}^d \to \mathbb{R}^d$ with

$$g(x) = rAx + z \quad \text{with } r \in (0,\infty),\ A \in O(d),\ z \in \mathbb{R}^d,$$

the most natural topology on G is possibly the topology induced by uniform convergence on bounded sets, or the topology induced by the norm topology on linear transformations.

On G, these are equivalent to the natural topology on $(0,\infty) \times O(d) \times \mathbb{R}^d$.
Another equivalent topology is the following:
Fix $\{x_0, \ldots, x_d\}$ in general position, let $\varepsilon > 0$ then for $g \in G$,

$$B(g,\varepsilon) := \{ f \in G \mid \|f(x_i) - g(x_i)\| < \varepsilon \ \forall i \}.$$

Finally, we have enough to state a thm:

Thm: (Bandt, Graf and Schief)

Let Φ be a self-similar IFS and let $s = \text{sim-dim}(\Phi)$. Then TFAE:

i.) Condition BG1

ii.) Condition BG2

iii.) Condition BG3

iv) SOSC

v.) OSC

vi.) $0 < \mathcal{H}^s(\Lambda)$

vii.) Λ is s-Ahlfors regular.

Plan

$$
\begin{tikzcd}
 & & \text{vii} \arrow[dl, Rightarrow] & \text{vi} \arrow[l, Rightarrow] \\
\text{BG1} \arrow[r, Leftrightarrow] & \text{BG3} \arrow[dr, Rightarrow] & & \\
\text{BG2} \arrow[ur, Leftrightarrow] & & \text{SOSC} \arrow[r, Rightarrow] & \text{OSC} \arrow[uu, Rightarrow]
\end{tikzcd}
$$

First, we prove

BG3 $\Rightarrow$ SOSC $\Rightarrow$ OSC $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ BG3.

1) BG3 $\Rightarrow$ SOSC.

pf: Let $\varepsilon > 0$ such that $(1+2\varepsilon) r_{max} < 1$

and

$$\sup_{\underline{k} \in \Sigma_1^*} \# \tilde{\Pi}_\varepsilon(\underline{k}) < \infty$$

Let $\underline{k}_0 \in \Sigma_1^*$ be such that

$$\# \tilde{\Pi}_\varepsilon(\underline{k}_0) = \max_{\underline{k} \in \Sigma_1^*} \# \tilde{\Pi}_\varepsilon(\underline{k}) .$$

Note that $\tilde{\Pi}_\varepsilon$ has the following stability condition:

$$\tilde{\Pi}_\varepsilon(\underline{i}\,\underline{j}) \supset \{ \underline{i}\,\underline{k} \mid \underline{k} \in \tilde{\Pi}_\varepsilon(\underline{j}) \}.$$
$$= \underline{i} \cdot \tilde{\Pi}_\varepsilon(\underline{j}).$$

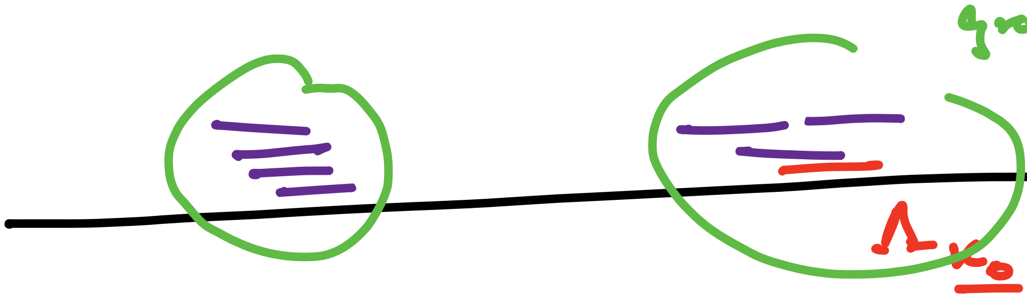
The maximality of $\underline{k}_0$ implies for any $\underline{i} \in \Sigma_1^*$

$$\tilde{\Pi}_\varepsilon(\underline{i}\,\underline{k}_0) = \underline{i} \cdot \tilde{\Pi}_\varepsilon(\underline{k}_0) .$$

Thus any j s.t. $j \in \tilde{\Gamma}_\varepsilon(\underline{i}\,\underline{k_0})$

$$\left(\begin{array}{l} \operatorname{diam}(\Lambda_j) \sim \phi_{\underline{i}\,\underline{k_0}}(N_\varepsilon(\Lambda)) \\ \text{and} \quad \Lambda_j \cap \phi_{\underline{i}\,\underline{k_0}}(N_\varepsilon(\Lambda)) \neq \emptyset \end{array} \right).$$

j must have $\underline{i}$ as a prefix.



group adjacent pieces and treat them like the IFS.

$\Rightarrow$ IF $j_1 \neq i_1$ then

$$\operatorname{dist}(\Lambda_{j_1}, \Lambda_{\underline{i}\,\underline{k_0}}) \geq \varepsilon\, r_{\underline{i}\,\underline{k_0}}$$

Now

$$V := \bigcup_{\underline{i} \in \Sigma^*} \phi_{\underline{i}\,\underline{k_0}}(N_{\varepsilon/2}(\Lambda))$$

satisfies the SOSC $\square$