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Similarity Dimension.

Let $\Phi = \{\phi_1, \dots, \phi_m\}$ be an IFS.

Let Λ be the attractor to Φ.

For $n \in \mathbb{Z}_+$, $\Sigma'_n := \prod_{k=1}^{n} \{1, \dots, m\}$

$\{\phi_{\underline{i}}(\Lambda)\}_{\underline{i} \in \Sigma'_n} = \{\Lambda_{\underline{i}}\}$ is a cover of Λ.

$$\operatorname{diam}(\phi_{\underline{i}}(\Lambda)) = \operatorname{diam}(\Lambda_{\underline{i}}) \leq \left(\prod_{j=1}^{n} r_{i_j}\right) \cdot \operatorname{diam}(\Lambda).$$

Then for $s > 0$,

$$\sum_{\underline{i} \in \Sigma'_n} \operatorname{diam}(\Lambda_{\underline{i}})^s \leq \sum_{\underline{i} \in \Sigma'_n} \operatorname{diam}(\Lambda)^s \, r_{i_1}^s r_{i_2}^s \cdots r_{i_n}^s$$

$$= \operatorname{diam}(\Lambda)^s \left(r_1^s + r_2^s + \dots + r_m^s\right)^n$$

If $\sum_{j=1}^{m} r_j^s < 1 \Rightarrow \sum_{\underline{i} \in \Sigma_n} \operatorname{diam}(\Delta_{\underline{i}})^s \xrightarrow{n \to \infty} 0$

$\Rightarrow \dim_{\mathcal{H}}(\Delta) \leq s.$

Since $g(s) = \sum_{j=1}^{m} r_j^s$ is decreasing, continuous,

$\dim_{\mathcal{H}}(\Delta) \leq \inf \{ s > 0 \mid \sum_{j=1}^{m} r_j^s < 1 \})$

$= s_{\Phi}$

where s_{Φ} satisfies $\sum_{j=1}^{m} r_j^{s_{\Phi}} = 1$

We define

s_{Φ} = similarity dimension of Φ

= sim-dim(Φ).

Examples

If $r_1 = r_2 = \cdots = r_m$

and if $s = \text{sim-dim}(\Phi)$ then

$m r^s = 1 \Rightarrow s = \dfrac{\log(m)}{\log(\frac{1}{r})}.$

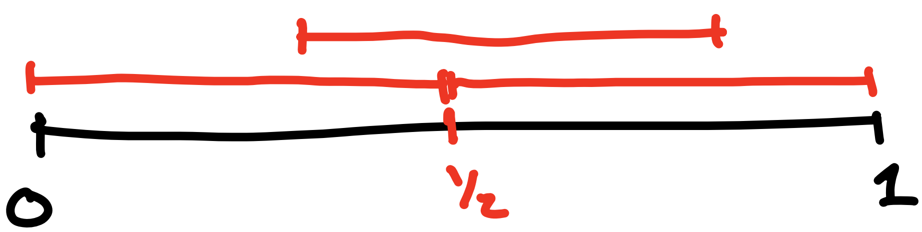
Q: When is $\text{sim-dim}(\Phi) = \dim_{\mathcal{H}}(\Lambda)$?

A: Not always.

Extreme example:

Let $\Phi = \{\phi_1, \phi_2, \phi_3\}$, $\phi_j : \mathbb{R} \to \mathbb{R}$.

$\phi_1(x) = \frac{1}{2}x$, $\phi_2(x) = \frac{1}{2}x + \frac{1}{2}$, $\phi_3(x) = \frac{1}{2}x + \frac{1}{3}$.



If Λ is the attractor of Φ, then

$$\dim_{\mathcal{H}}(\Lambda) = 1 \neq \frac{\log(3)}{\log(2)} = \text{sim-dim}(\Phi).$$

If we establish separation conditions, we can guarantee the equivalence of similarity and Hausdorff dimension.

Definition

Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be an IFS in $\mathbb{R}^d$ and Λ its attractor

a.) The <u>Strong Seperation Property</u> (SSP) holds for Φ if

$$\phi_i(\Lambda) \cap \phi_j(\Lambda) = \emptyset \quad \forall i \neq j$$

b.) The <u>Open Set Condition</u> (OSC) holds for Φ if there exists a nonempty bounded open set $V \subset \mathbb{R}^d$ such that

- $\phi_i(V) \subset V \quad \forall i \in \{1, \ldots, m\}$
- $\phi_i(V) \cap \phi_j(V) = \emptyset \quad \forall i \neq j$

c.) The <u>Strong Open Set Condition</u> (SOSC) holds for Φ if the set V is the definition of the OSC can be chosen so that

$$V \cap \Lambda \neq \emptyset.$$

About the Open Set Condition

An interesting property of the open set condition is that it is no always easy to determine the open set that satisfies the condition.

For the four corner Cantor set and the middle-thirds Cantor set, one can take

$$V = \mathrm{int}(\mathrm{conv}(\Delta))$$ where E is the attractor,

but the open set need not be connected. For example, take $0 < \frac{1}{M} < r << 1 << M$

$\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$

$$\phi_1(x) = rx + M$$
$$\phi_2(x) = rx + M + 1$$

Claim: (ϕ_1, ϕ_2) satisfy the open set condition but that open set is not connected

If V were a connected open set satisfying the open set condition, then if $\xi \in V$

$\phi_2(\xi) \in V$, $\Rightarrow$ $r\xi + M \in M$.

and $|\phi_2(\xi) - \xi| > M$. So V has an interval of length M contained inside of it. Now in order to violate $\phi_1(V) \cap \phi_2(V) = \phi$ it suffices to find $x, y \in V$ s.t.

$$rx = ry + 1 \iff x - y = \frac{1}{r} < M$$

So we are done.

Thm: (Moran '46, Hutchinson '81)
Assume that the self-similar IFS, $\Phi = (\phi_1, \dots, \phi_m)$ acts on $\mathbb{R}^d$ and satisfies the SSP. Then, if $s = \text{sim-dim}(\Phi)$,

$$0 < \mathcal{H}^s(\Lambda) < \infty \quad \text{and}$$

$$\dim_{\mathcal{H}}(\Lambda) = \dim_B(\Lambda) = s.$$

Pf:
We prove the statement in two steps:

1) $0 < \mathcal{H}^s(\Lambda) < \infty$ ($\Rightarrow \dim_{\mathcal{H}}(\Lambda) \geq s$)

2) $\overline{\dim}_B(\Lambda) \leq s$ (Now $s \leq \dim_{\mathcal{H}}(\Lambda) \leq \underline{\dim}_B \leq \overline{\dim}_B \leq s$).

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1) $$\mathcal{H}^s_\delta(\Lambda) \leq \limsup_{n\to\infty} \sum_{\underline{i} \in \Sigma_n} \operatorname{diam}(\Lambda_{\underline{i}})^s$$

$$\leq \limsup_{n\to\infty} \operatorname{diam}(\Lambda)^s \left(\sum_{j=1}^m r_j^s \right)^n$$

$$= \operatorname{diam}(\Lambda)^s < \infty \qquad \text{for all } \delta$$

$\Rightarrow \quad \mathcal{H}^s(\Lambda) < \infty.$

In order to prove $\mathcal{H}^s(\Lambda) > 0$, we define a measure on $\Sigma_1^1 = \{1, \dots, m\}^{\mathbb{N}}$.

<u>Recall</u>: $[i_1 \cdots i_n] = \left\{ \underline{j} \in \Sigma_1^1 \;\middle|\; j_k = i_k \text{ for } k = 1, \dots, n \right\}.$

Define

$$\mu([i_1 \cdots i_n]) = r_{i_1}^s \cdots r_{i_n}^s$$

Then $\displaystyle\sum_{\underline{i} \in \Sigma_n} \mu([\underline{i}]) = (r_1^s + \dots + r_m^s)^n = 1.$

μ can be extended to a Borel measure on Σ_1^1.

Recall: Natural projection $\pi : \Sigma_1^1 \to \Lambda$

Since Φ satisfies SSP, π is a continuous bijection.

Define the push-forward measure

$$\nu(A) := \mu(\pi^{-1}(A))$$

Thus, for $\underline{i} \in \Sigma_n'$

$$\nu(\Lambda_{\underline{i}}) = \mu([\underline{i}]) = r_{i_1}^s \cdots r_{i_n}^s$$

The SSP implies that

$$\ell := \min \{ \operatorname{dist}(\Lambda_i, \Lambda_j) \mid i \neq j \} > 0.$$

Then, for $\underline{i} \in \Lambda_n$

$$\min \{ \operatorname{dist}(\Lambda_{\underline{i}j}, \Lambda_{\underline{i}k}) \mid k \neq j \} = \ell r_{i_1} \cdots r_{i_n} \quad (*)$$

Now it suffices to show $\exists C > 0$ s.t.

$$\nu(B(x,r)) \leq C r^s$$

for all $x \in \Lambda$ and r small enough.

Fix $x \in \Lambda$, $r < \ell$, and let $\underline{i} = \pi^{-1}(x)$.

Let n be such that

$$\ell \, r_{i_1} \cdots r_{i_n} \leq r \leq r_{i_1} \cdots r_{i_{n-1}} \ell$$

$(*)$ implies

$$B(x,r) \cap \Lambda = B(x,r) \cap \Lambda_{i_1 \cdots i_{n-1}}.$$

Thus

$$\nu(B(x,r))$$

$$\leq \nu(\Delta_{i_2 \cdots i_{n-1}})$$

$$\leq r_{i_2}^s \cdots r_{i_{n-1}}^s$$

$$\leq \left(r_{min}^{-s}\right) r_{i_2}^s \cdots r_{i_{n-1}}^s$$

$$\leq (\rho\, r_{min})^{-s} \; r^s .$$