Easy Computations of Hausdorff and Packing dimension

(Hata '86)

Thm: Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers, and $\{r_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers strictly less than one.

Let $\Sigma'_N := \prod_{k=1}^{N} \{1, \ldots, n_k\}$.

Let $\Sigma'^{*} := \bigcup_{N=1}^{\infty} \Sigma'_N \cup \phi$

A define a function from Σ'^{*} to the set of closed balls in $\mathbb{R}^d$ by

$$B : \Sigma'^{*} \to \text{Closed balls.}$$

so that B satisfies

i.) $B(\phi) = \overline{B(0,1)}$

ii.) For $\omega_{N+1} \in \{1, \ldots, n_{N+1}\}$, $\underline{\omega} \in \Sigma'_N$

$$B(\underline{\omega}\, \omega_{N+1}) \subset B(\underline{\omega}).$$

iii.) For $\underline{\omega}_1 \in \Sigma'_N$ and $\underline{\omega}_2 \in \Sigma'_N$,

$$B(\underline{\omega}_1) \cap B(\underline{\omega}_2) = \phi.$$

iv.) For all $\underline{\omega} \in \Sigma_N$,

$$r_N = \mathrm{diam}(B(\underline{\omega})).$$

Now let $E_N := \bigcup_{\underline{\omega} \in \Sigma_N} B(\underline{\omega})$

and $E := \bigcap_{N=1}^{\infty} E_N$

Then

$$\underline{\dim}_B(E) = \dim_H(E) = \liminf_{N\to\infty} \frac{\log\left(\prod_{k=1}^N n_k\right)}{\log(r_N^{-1})}.$$

and

$$\overline{\dim}_B(E) = \dim_P(E) := \limsup_{N\to\infty} \frac{\log\left(\prod_{k=1}^N n_k\right)}{\log(r_N^{-1})}$$

Pf: From the corollary to the packing dimension proposition.

For any open, U.

$$\overline{\dim}_B(E) = \overline{\dim}_B(E \cap U) = \dim_P(E)$$

so it suffices to show the first equation.

Note that

$$d_{\mathcal{H}}(N_{r_N}(E), E_N) \sim r_N$$ where

$N_\delta(E)$ is the δ-neighborhood of E

and $d_{\mathcal{H}}$ is the Hausdorff distance

Then $$\underline{\dim}_B(E) = \liminf_{N\to\infty} \frac{\log\left(\prod_{k=1}^{N} n_k\right)}{\log(r_N^{-1})}.$$

and thus

$$\dim_{\mathcal{H}}(E) \leq \underline{\dim}_B(E) = \liminf_{N\to\infty} \frac{\log\left(\prod_{k=1}^{N} n_k\right)}{\log(r_N^{-1})}.$$

It suffices to show the reverse inequality.

Let $\mathcal{F} = \{\text{Borel sets}\}$

Define a sequence of functions, $T_N : \mathcal{F} \to \mathbb{N}$ by

$$T_N(F) = \#\{ \underline{\omega} \in \Sigma'_N \mid F \cap B(\underline{\omega}) \neq \emptyset \}.$$

Note that

$$T_{N+1}(F) \le n_{N+1} T_N(F)$$

Now define

$$\psi(F) := \lim_{N\to\infty} \frac{T_N(F)}{n_2 \cdots n_N}$$

Is the limit of decreasing functions, so it is well-defined.

ψ is subadditive and monotonic since each T_N is subadditive and monotonic.

Since $T_N(F) \le \prod_{k=1}^{N} n_k$ for each N,

$\psi(F) \le 1$ for all $F \in \mathcal{F}$ and $\psi(E)=1$.

Also, for $B(x,r)$, $r < \frac{1}{2} r_N$, $T_N(B(x,r)) \le 1$.

Let $$\gamma := \liminf_{N\to\infty} \frac{\log\left(\prod^{N} n_k\right)}{\log(r_N^{-1})} > 0.$$

For $\delta \in (0,\gamma)$ $\exists N_\delta \in \mathbb{N}$ s.t.

for $n \ge N_\delta$, $r_N^{\gamma-\delta} \prod_{k=1}^{N-1} n_k \ge 1$ $\left(\text{Since } \frac{\log(n_N)}{\log(r_N^{-1})} < \infty \right)$

Consider an arbitrary closed ball, $B \in \mathcal{F}^{\mu}$, with $\psi(B) > 0$, then there exists

Since $\psi(B) \le 1$, $\exists M \ge 1$ s.t.

$$\left(\prod_{k=1}^{M-1} n_k\right)^{-1} \ge \psi(B) \ge \left(\prod_{k=1}^{M} n_k\right)^{-1}.$$

(where $\prod_{k=1}^{0} n_k := 1$).

If $\operatorname{diam}(B) < r_M$, then

$$\psi(B) \le \frac{T_M(B)}{\prod^{M} n_k} \le \frac{1}{\prod_{k=1}^{M} n_k}$$ which is not possible

Thus, $\operatorname{diam}(B) \ge r_M$, then

$$\psi(B) \le \left(\prod_{k=1}^{M-1} n_k\right)^{-1} \le \left(\frac{\operatorname{diam}(B)}{r_M}\right)^{\gamma-\delta} \left(\prod^{M-1} n_k\right)^{-1}$$

$$\le (\operatorname{diam}(B))^{\gamma-\delta}$$

Finally, for a finite cover of E by closed balls, $\{B_m\}$

$$1 = \psi(E) \le \psi\Big(\bigcup_m B_m\Big)$$

$$\le \sum_m \psi(B_m) \le \sum_m \operatorname{diam}(B)^{\gamma-\delta}$$

$$\Rightarrow \quad 1 \le \mathcal{H}^{\gamma-\delta}(E) \quad \text{for all } \delta \in (0,\gamma).$$

$\square$.

The Duality of Hausdorff and Packing Dimension

<u>Thm</u>: Let $E \subset \mathbb{R}^d$, $F \subset \mathbb{R}^n$

Then

$$\dim_{\mathcal{H}}(E) + \dim_{\mathcal{H}}(F) \leq \dim_{\mathcal{H}}(E \times F) \leq \dim_{\mathcal{H}}(E) + \dim_{p}(F).$$

<u>pf</u>: Suppose $\mathcal{H}^s(E) > 0$ and $\mathcal{H}^t(F) > 0$

Then Frostman's Lemma

$\Rightarrow$ $\exists\, \mu, \nu$ s.t. $\mathrm{supp}(\mu) \subset E$
$\mathrm{supp}(\nu) \subset F$

and $\mu(A) \lesssim (\mathrm{diam}(A))^s$

and $\nu(B) \lesssim (\mathrm{diam}(B))^t$

Then $\mu \times \nu$ is supported in $E \times F$ and

$$\mu \times \nu (U) \leq \mu \times \nu (P_1 U \times P_2 U)$$

$$= \mu(P_1 U)\, \nu(P_2 U)$$

$\lesssim (\operatorname{diam} U)^{s+t}$

Mass distribution principle

$\Rightarrow \mathcal{H}^{s+t}(E \times F) > 0$

and $\dim_{\mathcal{H}}(E \times F) > s+t$.

For $\dim_{\mathcal{H}}(E \times F) \leq \dim_{\mathcal{H}}(E) + \dim_P(F)$, it

It suffices to show that If $F = \bigcup_{j=1}^{\infty} F_j$

$$\dim_{\mathcal{H}}(E \times F_j) \leq \dim_{\mathcal{H}}(E) + \overline{\dim}_B(F_j).$$

Let $s > \dim_{\mathcal{H}}(E)$ and $t > \overline{\dim}_B(F_j)$

Then $\mathcal{H}^s(E) = 0$ and

$$2^{-kt} \# \{ q \in \mathcal{D}_k \mid q \cap F_j \neq \emptyset \} \xrightarrow{k \to \infty} 0$$

Let $\delta_0 > 0$ be such that

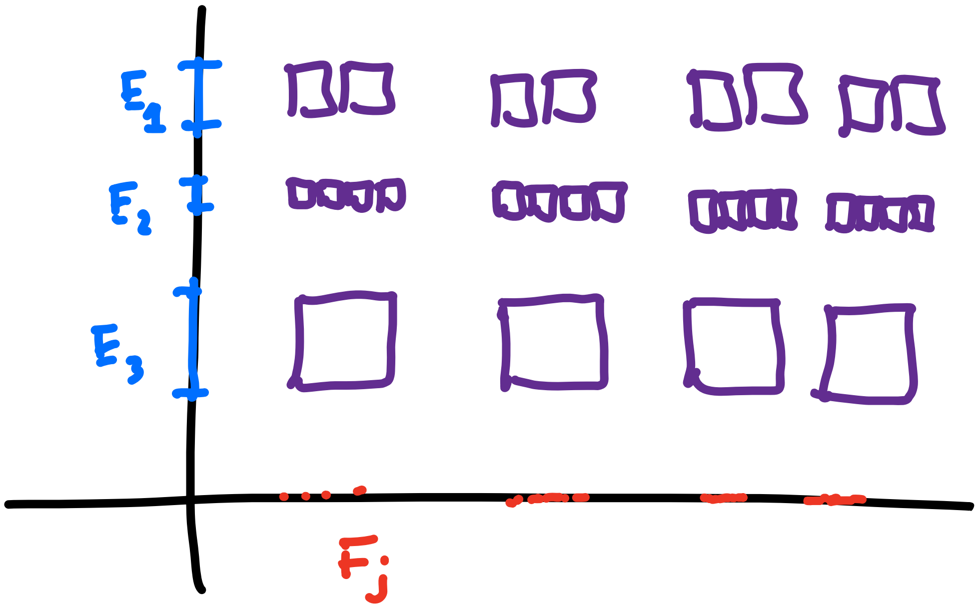
for $2^{-k} < \delta_0$

$$2^{-kt} \# \{ q \in \mathcal{D}_k \mid q \cap F_j \neq \emptyset \} < 1 \quad \text{and}$$

and for $\delta < \delta_0$ $\exists \{E_i\}_{i=1}^{\infty}$ such that $E \subset \bigcup_i E_i$ and $\sum (\operatorname{diam}(E_i))^s < 1$.

Let $$Q_{k_i} := \{ q \in \mathcal{D}_{k_i} \mid q \cap F_j \neq \emptyset \}.$$

Then where $2^{-k_i} < \delta_0$ and $2^{-k_i} \sim \operatorname{diam}(E_i)$.



$$\mathcal{H}^{s+t}_{100\delta}(E \times F_j) \leq \sum_{i,\, q \in Q_{k_i}} \operatorname{diam}(E_i \times q)^{s+t}.$$

$$\lesssim_d \sum_i \operatorname{diam}(E_i)^s \Big(\sum_{q \in Q_{k_i}} \operatorname{diam}(q)^t \Big)$$

$$< \sum_i \operatorname{diam}(E_i)^s < 1$$

$$\Rightarrow \mathcal{H}^{s+t}(E \times F_j) < \infty$$

$\square$.