# Packing Measure and Dimension

Let $(X,\rho)$ be a metric space.

For $E \subset X$, a $\underline{\delta\text{-packing}}$ of $E$ is a collection of disjoint closed balls, $\{\overline{B(x_i,r_i)}\}$ such that $x_i \in E$, $0 < r_i \le \delta$ for all $i$.

For $s > 0$, let

$$\tilde{\mathcal{P}}^s_\delta(E) := \sup\left\{ \sum_i |\overline{B}_i|^s \;\middle|\; \begin{array}{l}\{\overline{B}_i\} \text{ is a} \\ \delta\text{-packing of } E\end{array} \right\}.$$

Now letting

$$\tilde{\mathcal{P}}^s(E) := \lim_{\delta \to 0} \mathcal{P}^s_\delta(E) = \inf_{\delta > 0} \mathcal{P}^s_\delta(E).$$

$\tilde{\mathcal{P}}^s$ is the $s$-dimensional packing pre-measure.

<u>Note:</u> $\tilde{\mathcal{P}}^s$ is not countably sub-additive.

<u>Example:</u> $\tilde{\mathcal{P}}^1(\mathbb{Q} \cap [0,1]) = 1$.

The s-dimensional Packing measure is now defined by

$$\mathcal{P}^s(E) := \inf\left\{ \sum_{j=1}^{\infty} \tilde{\mathcal{P}}^s(E_j) \;\middle|\; E \subset \bigcup_{j=1}^{\infty} E_j \right\}.$$

The Packing dimension is defined by

$$\dim_P(E) := \inf\{ s > 0 \mid \mathcal{P}^s(E) = 0 \}.$$

Comparison with Hausdorff and Box-counting dimensions.

$$\dim_H(E) \le \dim_P(E) \le \overline{\dim}_B(E).$$

Why?

For $\dim_P(E) \le \overline{\dim}_B(E)$

Since covering by dyadic cubes constitutes a packing, this is clear.

Let $\varepsilon > 0$ and

$$s = \overline{\dim}_B(E) + \varepsilon.$$

Then

$$\overline{\lim_{k\to\infty}} \frac{\log\left(\#\{q \in \mathcal{D}^k \mid q \cap E \neq \phi\}\right)}{k \log 2} \leq s - \varepsilon/100$$

$\Rightarrow$ For $k$ large enough

$$2^{-ks} \cdot \#\{q \in \mathcal{D}^k \mid q \cap E \neq \phi\} \leq 2^{-\varepsilon/100\, k}.$$

It is clear that

$$\tilde{\mathcal{P}}^s_\delta(E) \leq (2^{-k})^s \,\#\{q \in \mathcal{D}^k \mid q \cap E \neq \phi\} \leq 2^{-\varepsilon/100\, k}.$$

For $\delta < 2^{-k}$

$\Rightarrow$

$$\mathcal{P}^s(E) \leq \hat{\mathcal{P}}^s(E) = 0.$$

$\Rightarrow$ $\dim_P(E) \leq s = \overline{\dim}_B(E) + \varepsilon$

$\forall \varepsilon > 0$. $\square$

In fact, we have the following proposition

Prop:

$$\dim_P(F) = \inf \left\{ \sup_{j \in I} \overline{\dim}_B(F_j) \;\middle|\; \begin{array}{l} F \subset \bigcup F_j \\ I \text{ is countable} \\ F_j \text{ are bounded} \end{array} \right\}$$

Cor:

Let $F \subset \mathbb{R}^d$ be compact.

If for all open $U \subset \mathbb{R}^d$ for which $F \cap U \neq \emptyset$ we have $\overline{\dim}_B(F \cap U) = \overline{\dim}_B(F)$

Then $\dim_P(F) = \overline{\dim}_B(F)$

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This corollary is very useful when $F$ is a self-similar set. We immediately get that $\dim_P(F) = \overline{\dim}_B(F)$.

## Proof of Prop

To prove the proposition, it is enough to show that for any $\varepsilon > 0$

$\exists \{F_j\}$ s.t. $F = \bigcup F_j$ and

$$\overline{\dim}_B (F_j) \le \dim_P (F) + \varepsilon \quad \forall j.$$

Let $s = \dim_P(F) + \varepsilon$. Then

$$\mathcal{P}^s(F) = 0$$

$\Rightarrow \exists \{F_j\}, \exists \delta_0 > 0$ s.t. $\bigcup_j F_j = F$ and

$\tilde{\mathcal{P}}_\delta^s(F_j) < 100$ for $\delta < \delta_0$. for all $j$.

For $2^{-k} < \delta_0$

$$(2^{-k})^s \cdot \#\{q \in \mathcal{D}^k \mid q \cap F_j \ne \phi\} < 100$$

$$\Rightarrow \frac{\log(\#\{q \in \mathcal{D}^k \mid q \cap F_j \ne \phi\})}{k \log 2} < \frac{\log(100) + sk\log 2}{k \log 2}$$

$$\Rightarrow \overline{\lim_{k\to\infty}} \frac{\log(\#\{q \in \mathcal{D}^k \mid q \cap F_j \ne \phi\})}{k \log 2} \le s.$$

For $\dim_{\mathcal{H}}(E) \le \dim_{p}(E)$

It suffices to show

$$\mathcal{H}^s(E) \le \mathcal{P}^s(E)$$

but since $\mathcal{H}^s$ is countably additive, it suffices to show that

$$\mathcal{H}^s(E) \le \tilde{\mathcal{P}}^s(E).$$

WLOG assume $\tilde{\mathcal{P}}^s(E) < \infty$.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\tilde{\mathcal{P}}^s_\delta(E) < \tilde{\mathcal{P}}^s(E) + \varepsilon.$$

Let $\{B_i\}_{i=1}^{\infty}$ be a collection of disjoint closed balls such that

$$\sum \operatorname{diam}(B_i)^s \le \tilde{\mathcal{P}}^s_\delta(E) \le \sum \operatorname{diam}(B_i)^s + \varepsilon.$$

$\exists\, K$ s.t.

$$\sum_{i=K+1}^{\infty} \operatorname{diam}(B_i)^s < \varepsilon.$$

Consider the family of closed balls

$$\mathcal{F} := \left\{ B(x,r) \;\middle|\; \begin{array}{l} B(x,r) \subset \mathbb{R}^d \setminus \bigcup_{i=1}^{k} B_i, \quad r < \frac{\delta}{100} \\ x \in E \end{array} \right\}.$$

5r covering lemma implies

$\exists$ disjoint $\{B_m'\}_{m=1}^{\infty}$ s.t.

$$E \setminus \bigcup_{i=1}^{k} B_i \subset \bigcup_{B(x,r)\in\mathcal{F}} B(x,r) \subset \bigcup_{k=1}^{\infty} 5B_m'.$$

Since $\{B_i\}_{i=1}^{k} \cup \{B_m'\}$ is a packing

$$\sum^{k} (\operatorname{diam}(B_i))^s + \sum \operatorname{diam}(B_m')^s$$

$$\leq \tilde{\mathcal{P}}_\delta^s(E) \leq \sum_{i=1}^{\infty} \operatorname{diam}(B_i)^s + \varepsilon.$$

$$\leq \sum_{i=1}^{k} \operatorname{diam}(B_i)^s + 2\varepsilon.$$

$$\Rightarrow \sum \operatorname{diam}(B_m')^s < 2\varepsilon.$$

$$\Rightarrow \mathcal{H}_\delta^s(E) \leq \sum_{i=1}^{k} \operatorname{diam}(B_i)^s + \sum \operatorname{diam}(5B_m')^s$$

$$\leq \tilde{\mathcal{P}}_\delta^s(E) + 5^s 2\varepsilon \leq \tilde{\mathcal{P}}^s(E) + C\varepsilon$$

$\square$.