Application to Falconer's Distance set Problem

Thm: (Falconer) Let $A \subset \mathbb{R}^d$ be Borel,

$$\dim_{\mathcal{H}}(A) > \frac{d+1}{2} \Rightarrow \mathcal{L}^1(\Delta(A)) > 0$$

(Here $\Delta(A) = \{\|x-y\| \mid x, y \in A\}$.).

Pf: WLOG assume $\operatorname{diam}(A) \le R$.
$\Rightarrow \Delta(A) \subset [0,R]$. Assum $\dim_{\mathcal{H}}(A) > \frac{d+1}{2}$.

Frostman's Lemma implies that there exists μ supported on A s.t.

$$\mathcal{E}_{\frac{d+1}{2}}(\mu) < \infty \quad \text{and} \quad \mu(A) = 1.$$

Suppose the intervals $\{I_j\}_{j=1}^{\infty}$ cover $\Delta(A)$. Then

$$A \times A \subset \bigcup_{j=1}^{\infty} \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid \|x-y\| \in I_j\}.$$

Then

$$1 = \mu\times\mu(A \times A) \le \sum \mu\times\mu(\{(x,y) \mid \|x-y\| \in I_j\})$$

Claim:

$$\mu\times\mu(\{(x,y) \mid \|x-y\| \in I_j\}) \lesssim R^{(d-1)/2}\, \mathcal{E}_{\frac{d+1}{2}}(\mu) \cdot \operatorname{diam}(I_j).$$

Assuming the claim holds

$$\frac{1}{R^{d-1/2}\, \mathcal{E}_{\frac{d+1}{2}}(\mu)} \lesssim \sum \operatorname{diam}(I_j)$$

for any cover of $\Delta(A)$

$$\Rightarrow \quad \mathcal{L}^1(\Delta(A)) \gtrsim \frac{1}{R^{d-1/2}\, \mathcal{E}_{\frac{d+1}{2}}(\mu)}.$$

Proof of Claim.

Let For $I = (r, r+\varepsilon)$

$$g = \chi_{\{x \in \mathbb{R}^d \mid |x| \in I\}}.$$

Then

$$\mu \times \mu (\{ (x,y) \mid |x-y| \in I \})$$

$$= \iint g(x-y)\, d\mu(x)\, d\mu(y).$$

$$= \langle g, \mu * \mu \rangle$$

$$= \langle \hat{g}, |\hat{\mu}|^2 \rangle.$$

Now $|\hat{g}(\xi)| \lesssim \min \begin{pmatrix} r^{(d-1)/2} |\xi|^{-(d-1)/2} \varepsilon, \\ r^{(d-1)/2} |\xi|^{-(d+1)/2} \end{pmatrix}$

$\Longrightarrow$

$$\mu \times \mu (\{ |x-y| \in I \})$$

$$\lesssim \varepsilon\, r^{(d-1)/2} \int |\xi|^{-(d-1)/2} |\hat{\mu}(\xi)|^2 d\xi.$$

$$= c\, \varepsilon\, r^{(d-1)/2} \iint |x-y|^{-(d+1)/2} d\mu(x)\, d\mu(y).$$

(Here we use $\widehat{|\cdot|^{-s}} = |\cdot|^{s-d}$).

With the Riesz energy, one can define capacitory dimension.

<u>Def:</u> Let μ be a Borel measure on $\mathbb{R}^d$, then the capacitary dimension of μ is

$$\dim_c(\mu) := \sup\{s \mid \mathcal{E}_s(\mu) < \infty\}.$$

Application to Marstrand Projection Thm

Q: Let $E \subset \mathbb{R}^d$, $\dim_{\mathcal{H}}(E) = s \in (0, d]$ and $m \in \{1, \dots, d\}$. Given $V \in G(d,m)$ what is the dimension of the orthogonal projection of E onto V, $P_V(E)$?

A: We will show that for almost every V, $\dim_{\mathcal{H}}(P_V(E)) = \min(m, s)$.

The "transversality" of the collection of projections will be vital.

Let Θ_d be the invariant probability measure on $O(d)$.

Fix $V_0 \in G(d,m)$. Define $\gamma_{d,m}$ by

$$\gamma_{d,m}(A) := \Theta_d\left(\{g \mid gV_0 \in A\}\right).$$

for $A \subset G(d,m)$

Lemma: $\exists c_d > 0$ s.t for any $x \in \mathbb{R}^d \setminus \{0\}$,

and $0 < \delta < \infty$,

$$\gamma_{d,m}(\{V \mid d(x,V) \le \delta\}) \le c_d \, \delta^{d-m} |x|^{m-d}$$

$$\gamma_{d,m}(\{V \mid |P_V(x)| \le \delta\}) \le c_d \, \delta^{m} |x|^{-m}.$$

Pf:

Note that

$$\gamma_{d,m}(A) = \gamma_{d,d-m}(\{V^\perp : V \in A\})$$

and $d(x, V^\perp) = |P_V x|$, so it suffices

to prove

$$\gamma_{d,m}(\{V \mid d(x,V) \le \delta\}) \lesssim \delta^{d-m} |x|^{m-d}$$

Note again that

$$d(x,V) = |x| \, d\left(\frac{x}{|x|}, V\right).$$

so we can assume $x \in S^{d-1}$.

Let $W = \{x \in \mathbb{R}^d \mid x_{m+1} = \cdots = x_d = 0\}$.

Now

$$\gamma_{d,m}(\{V \mid d(x,V) \le \delta\})$$

$$= \theta_d(\{g \in O(d) \mid d(x, gW) \le \delta\})$$

$$= \theta_d(\{g \mid d(g^{-1}x, W) \le \delta\})$$

surface measure on S^{d-1}

$$= \sigma^{d-1}(\{y \in S^{d-1} \mid d(y, W) \le \delta\})$$

$$= \sigma^{d-1}\left(\left\{ y \in S^{d-1} \,\middle|\, \left(\sum_{i=m+1}^{d} y_i^2\right)^{1/2} \le \delta \right\}\right)$$

$$\le c_d \mathcal{L}^d\left(\left\{ z \in \mathbb{R}^d \,\middle|\, \begin{array}{l} |z_i| \le 1 \text{ for } i \le m \\ \text{and } |z_i| \le \delta \text{ for } i > m \end{array} \right\}\right)$$

$$\lesssim \delta^{d-m}$$

$\square$.

Cor: For $s \in (0, m)$ $\exists\, c = c(d,m,s) > 0$ s.t. for $x \in \mathbb{R}^d \setminus \{0\}$

$$\int |P_V x|^{-s} \, \gamma_{d,m}(dV) \le c\,|x|^{-s}.$$

Pf:

$$\int |P_V x|^{-s} \gamma_{d,m}(dV) = \int_0^{\infty} \gamma_{d,m}(\{V \mid |P_V x|^{-s} \ge t\})\, dt.$$

$$= \int_0^\infty \gamma_{d,m}(\{V \mid |P_V x| < t^{-1/s}\})\, dt$$

$$= \int_0^{|x|^{-s}} dt + \int_{|x|^{-s}}^\infty \gamma_{d,m}(\{ |P_V x| < t^{-1/s}\})\, dt.$$

$$\lesssim |x|^{-s} + |x|^{-m} \int_{|x|^{-s}}^\infty t^{-\frac{m}{s}}\, dt.$$

$$\lesssim |x|^{-s}. \qquad \square$$

We now define the s-Riesz capacity by

$$C_s(A) := \sup\left\{ \mathcal{E}_s(\mu)^{-1} \;\middle|\; \begin{array}{l} \mu \in \mathcal{M}(A) \\ \mu(\mathbb{R}^d) = 1 \end{array} \right\}.$$

$[C_s(\emptyset) = 0)$.

$\left(\dim_c(A) := \sup\{ s \mid C_s(A) > 0\} \right.$ and $\left. \dim_c(A) \leq \dim_{\mathcal{H}}(A) \right)$.

Thm: Let $0<s<m$. $\exists\, c = c(m,d,s)>0$ s.t. for $A\subset\mathbb{R}^d$,

$$\int C_s(P_V A)^{-1}\,\gamma_{d,m}(dV) \le c\, C_s(A)^{-1}.$$

In particular, $C_s(A)>0 \Rightarrow$

$C_s(P_V A)>0$ for $\gamma_{d,m}$-a.e. $V\in G(d,m)$

Pf:

Let μ be a Radon measure with compact support such that $\operatorname{supp}\mu\subset A$ and $\mu(A)=1$.

Then $P_{V\#}\mu \subset P_V(A)$ and $P_{V\#}\mu(P_V A)=1$

$$\Rightarrow \quad C_s(P_V(A))^{-1} \le \mathcal{E}_s(P_{V\#}\mu).$$

$$\Rightarrow \int C_s(P_V A)^{-1}\,d\gamma \le \int \mathcal{E}_s(P_{V\#}\mu)\,d\gamma$$

$$= \int \left(\iint |P_V(x-y)|^{-s}\,\mu(dx)\,\mu(dy)\right) d\gamma$$

$$\doteq \iint \int |P_V(x-y)|^{-s}\,d\gamma \quad \mu(dx)\,\mu(dy)$$

$$\lesssim \iint |x-y|^{-s}\,\mu(dx)\,\mu(dy) = \mathcal{E}_s(\mu)$$

$\square$.