__Hausdorff dimension__. Let (X,ρ) be a metric space.

Let $E \subset X$, $s \geq 0$, $\delta > 0$. The Hausdorff pre-measure is the following set function:

$$\mathcal{H}^s_\delta(E) := \inf\left\{ \sum_{i=1}^{\infty} |A_i|^s \,\middle|\, E \subset \bigcup_{i=1}^{\infty} A_i,\ |A_i| \leq \delta \right\}$$

The Hausdorff measure is defined by

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

__Lemma:__ For any Borel set $E \subset \mathbb{R}^d$,

$$\mathcal{H}^d(E) = \frac{2^d \left(\frac{d}{2}\right)!}{\pi^{d/2}} \mathcal{L}^d(E)$$

__Hausdorff Dimension__

$$\dim_{\mathcal{H}}(E) := \inf\{ s \mid \mathcal{H}^s(E) = 0 \}$$
$$= \sup\{ s \mid \mathcal{H}^s(E) = \infty \}.$$

Lemma:

- Every countable set has zero Hausdorff dimension
- For every $F \subset \mathbb{R}^d$, $\dim_{\mathcal{H}}(F) \leq d$.
- If $\mathcal{L}^d(E) > 0$ for a Borel set, $E \subset \mathbb{R}^d$, then $\dim_{\mathcal{H}}(E) = d$.
- Let $f: X \to Y$ be Lipschitz and let $E \subset X$. Then
$$\dim_{\mathcal{H}}(f(E)) \leq \dim_{\mathcal{H}}(E)$$
$$(\mathcal{H}^s(f(E)) \leq c\, \mathcal{H}^s(E))$$
- Let $f: X \to Y$ be bi-Lip. Then
$$\dim_{\mathcal{H}}(E) = \dim_{\mathcal{H}}(f(E)).$$
- $$\dim_{\mathcal{H}}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sup_i \dim_{\mathcal{H}}(E_i).$$

Lemma: Let $f: X \to Y$ be α-Hölder. Then for $E \subset X$, $s \geq 0$,
$$\mathcal{H}^s(f(E)) \leq c^t\, \mathcal{H}^{s\alpha}(E).$$

and
$$\dim_H(f(E)) \le \frac{\dim_H(E)}{\alpha}.$$

Lemma:

Let $\Phi = (\phi_1, \ldots, \phi_m)$ be an IFS on $\mathbb{R}^d$, with the Lipschitz constants $0 < r_i < 1$ for $i \le m$. Let $\Pi : \Sigma_1 \to \Lambda$ be the natural projection corresponding to Φ and let $r_{max} := \max_i r_i$

Then Π is α-Hölder continuous with
$$\alpha = \frac{\log(r_{max}^{-1})}{\log m}.$$

Pf:

Let $\underline{i}, \underline{j} \in \Sigma_1$, let $\underline{\omega} = \underline{i} \wedge \underline{j}$ and let $n := |\underline{i} \wedge \underline{j}| \ge 0$. Then with s being the shift operator, we have

$$|\Pi(\underline{i}) - \Pi(\underline{j})| = |\Pi(\underline{\omega}\, s^n \underline{i}) - \Pi(\underline{\omega}\, s^n \underline{j})|$$

$$= \left| \phi_{\underline{\omega}} \circ \pi(s^n \underline{i}) - \phi_{\underline{\omega}} \circ \pi(s^n \underline{j}) \right|$$

$$\leq r_{max}^n \cdot \operatorname{diam}(\Lambda)$$

Note: $m^{\alpha} = r_{max}^{-1}$.

$$= \operatorname{diam}(\Lambda) \left(m^{-n}\right)^{\alpha}$$

$$= \operatorname{diam}(\Lambda)\, \rho(\underline{i}, \underline{j})^{\alpha}$$

$\square$.

Example: Hausdorff dimension of the middle-third cantor set.

Note: The Hausdorff dimension of (Σ_1^+, ρ) is 1.

Since $\underline{\Phi} = \left(\phi_1(x) = \frac{1}{3}x, \ \phi_2(x) = \frac{1}{3}x + 2/3 \right)$ the natural projection, $\pi : \Sigma_1^+ \to \Lambda$, is Hölder continuous with exponent

$$\alpha = \frac{\log(3)}{\log(2)}.$$

Thus if Λ = middle-third Cantor set,

$$\dim_H(\Lambda) = \dim_H(\pi(\Sigma_1)) \le \frac{1}{\alpha} = \frac{\log(2)}{\log(3)}.$$

To show the reverse direction, it suffices to show that any covering of Λ, $\{I_j\}$, satisfies

$$1 \le \sum_{j=1}^{\infty} |I_j|^s, \quad \text{where} \quad s = \frac{\log(2)}{\log(3)}.$$

Since Λ is compact, it suffices to assume $\{I_j\}$ is a finite collection.

We can assume that I_j covers two cylinders

$\Lambda_{i_1 \dots i_m}$ and $\Lambda_{i_1 \dots i_m j_1 \dots j_k}$.

Thus $I_j = I_{i_1\cdots i_m} \cup J \cup I_{i_1\cdots i_m j_1\cdots j_k}$

with $|I_{i_1\cdots i_m}|, |I_{i_1\cdots i_m j_1 \cdots j_k}| \le J$.

That

$$|I_j|^s = |I_{i_1\cdots i_m} \cup J \cup I_{i_1\cdots i_m j_1\cdots j_k}|^s$$

$$= \left(|I_{i_1\cdots i_m}| + |J| + |I_{i_1\cdots i_m j_1\cdots j_k}| \right)^s$$

$$\ge \left(\tfrac{3}{2} \left(|I_{i_1\cdots i_m}| + |I_{i_1\cdots i_m j_1\cdots j_k}| \right) \right)^s$$

$$2 = \quad = \boxed{3^s} \left(\tfrac{1}{2} |I_{i_1\cdots i_m}| + |I_{i_1\cdots i_m j_2\cdots j_k}| \right)^s$$

$$\ge 2 \left(\tfrac{1}{2} |I_{i_1\cdots i_m}|^s + \tfrac{1}{2} |I_{i_1\cdots i_m j_2\cdots j_k}|^s \right).$$

$$\ge |I_{i_1\cdots i_m}|^s + |I_{i_1\cdots i_m j_2\cdots j_k}|^s.$$

Thus the smallest covering consists of elements of the form $I_{i_1\cdots i_m}$.

Mass Distribution and Frostman Measures.

Lemma: (Mass Distribution Principle)
Let E be a Borel set in a metric space, and let μ be a positive finite Borel measure such that $\mu(E)>0$.
Assume that for some $s>0$, there exist $c>0$ and $\delta>0$ s.t.

$$\mu(A)\le c(\operatorname{diam} A)^s \quad \text{for } \operatorname{diam}(A)<\delta.$$

Then we have

$$\mathcal{H}^s(E)\ge \frac{\mu(E)}{c}$$

and $\dim_{\mathcal{H}}(E)\ge s$.

Pf: Straightforward.

Def: A compactly supported positive Borel measure μ on a metric space, X, is called a Frostman measure if $0<\mu(X)<\infty$ and

$$\mu(B(x,r))\le C r^s \quad \text{for all } x\in X,\ r>0.$$

Frostman's Lemma

If $X \subset \mathbb{R}^d$ is a compact set w/ $\mathcal{H}^s(X) \in (0,\infty)$, then $\exists$ finite Borel measure, μ, on X s.t. $\mu(X) > 0$ and for some $C > 0$,

$$\mu(B(x,r)) \leq C r^s \quad \text{for all } x \in X, \ r > 0.$$

Recall:

Egoroff's Thm Let $\mu(X) < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence converging μ-a.e. to a function, f. For any $\varepsilon > 0$, there exists a set $F \subset X$ s.t. $\mu(X \setminus F) < \varepsilon$ and

$$f_n|_F \to f|_F \text{ uniformly.}$$

Proof of Frostman Lemma.

Let $E = \left\{ x \in X \;\middle|\; \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s} \geq 100 \right\}$.

Then $\mathcal{H}^s(E) \leq C_* \frac{1}{100} \mathcal{H}^s(X)$.

Let $f_k : X \setminus E \to \mathbb{R}$.

$$f_k(x) = \sup_{r \leq 2^{-k}} \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s}$$

Egoroff $\Rightarrow$ $\exists$ $F, \delta > 0$ s.t.

$\mathcal{H}^s(F) \geq \frac{1}{2} \mathcal{H}^s(X)$ s.t.

$$\mathcal{H}^s(F \cap B(x,r)) \leq 100\,(2r)^s$$

For all $x \in F$ and $r < \delta$

For $r \geq \delta$, $\mathcal{H}^s(F \cap B(x,r))$

$$\leq \frac{\mathcal{H}^s(X)}{(2\delta)^s} \cdot (2r)^s.$$

Let $\mu := \mathcal{H}^s|_F$ $\square$.

Energies and Hausdorff Dimension.

Let $0 < \mu(\mathbb{R}^d) < \infty$, $s > 0$.

The <u>s-energy</u> of μ is defined by

$$\mathcal{E}_s(\mu) := \iint |x-y|^{-s} \, d\mu(x) \, d\mu(y)$$

<u>Lemma:</u> Suppose $0 < \mu(\mathbb{R}^d) < \infty$

IF μ is s-Frostman then

$$\mathcal{E}_t(\mu) < \infty \quad \text{for } t < s$$

<u>pf.</u> Simple computation. Let $I_t(\mu, x) = \int \frac{1}{|x-y|^t} d\mu$

$$\begin{aligned}
I_t(\mu, x) &= \int_0^\infty \mu(\{ |x-y|^{-t} \geq r \}) \, dr \\
&= \int_0^\infty \mu(B(x, r^{-1/t})) \, dr \\
&\lesssim \int_0^\infty \min(\mu(\mathbb{R}^d), C r^{-s/t}) \, dr \\
&\lesssim_\mu \int_1^\infty r^{-s/t} \, dr < \infty. \quad \square
\end{aligned}$$

Lemma:

If μ is a finite measure supported on E, then

$$\mathcal{E}_t(\mu) < \infty \Rightarrow \dim_{\mathcal{H}}(E) \geq t$$

Pf: $\mathcal{E}_t(\mu) = \int I_t(\mu, x)\, d\mu < \infty$.

$\Rightarrow$ $\exists M > 0$ and $E_M \subset \mathbb{R}^d$, s.t.

$$I_t(\mu, x) \leq M \quad \text{for all } x \in E_M$$

and $\mu(E_M) > 0$

Now for any $A \subset \mathbb{R}^d$, with $A \cap E_M \neq \phi$, and $x \in A \cap E_M$

$$M \geq I_t(\mu, x) = \int_A \frac{1}{|x-y|^t}\, d\mu|_{E_M}(y) \geq \mu|_{E_M}(A) \left(\operatorname{diam}(A)\right)^{-t}.$$

$$\Rightarrow \mu|_{E_M}(A) \leq M \left(\operatorname{diam}(A)\right)^t$$

$$\Rightarrow \mu(E_M) \lesssim \mathcal{H}^t(E_M)$$

$$\Rightarrow t \leq \dim_{\mathcal{H}}(E_M) \leq \dim_{\mathcal{H}}(E).$$

$\square$.