Distance Sets and Entropy

We would like to end with an application of the entropy methods of Hochman to Falconer's distance set conjecture. The following is a theorem of Orponen:

Thm (Orponen '15)

Assume $K \subset \mathbb{R}$ is a compact s-Ahlfors regular set with $s \geq 1$, then

$$\dim_P(\Delta(K)) = 1.$$

where $\Delta(K) = \{ \|x-y\| \mid x, y \in K \}$.

Idea

Recall that Falconer's proof uses the estimate

$$* \quad \mu\times\mu\left(\{(x,y) \mid \|x-y\| \in I_j\}\right) \lesssim R^{(d-1)/2}\, \mathcal{E}_{\frac{d+1}{2}}(\mu)\cdot \operatorname{diam}(I_j).$$

where $\{I_j\}_{j=1}^{M}$ is an interval cover of $\Delta(K)$.

$$\Rightarrow \quad \mathcal{L}^1(\Delta(K)) \gtrsim \frac{1}{\mathcal{E}_{\frac{d+1}{2}}(\mu)}$$

$*$ Follows from the following estimate:

$$\mu\times\mu\left(\{(x,y) \mid \|x-y\| \in I_j\}\right) = \iint \chi_{\{\|\cdot\|\in I_j\}}(x-y)\, d\mu(x)\, d\mu(y)$$

$$= \int \widehat{\chi}_{\{\|\cdot\|\in I_j\}}(\xi)\, |\widehat{\mu}(\xi)|^2\, d\xi.$$

$$\widehat{\chi}_{\{\|\cdot\| \in I_j\}}(\xi) \sim \min\left(|I_j|, |\xi|^{-1}\right) \widehat{\chi}_{S^{d-1}}(\xi)$$

$$\widehat{\chi}_{S^{d-1}}(\xi) \lesssim |\xi|^{-(d-1)/2}$$

$$\Rightarrow \quad \mu \times \mu(\{\quad\}) \lesssim \int \min\left(|I_j|, |\xi|^{-1}\right) |\xi|^{-(d-1)/2} |\hat{\mu}(\xi)|^2 \, d\xi$$

$$\lesssim |I_j| \; I_{\frac{d+1}{2}}(\mu).$$

Orponen's result starts with the following observation: Fixing $x \in K$, let

$$f_x(y) = \|x-y\|^2 / \|x\|$$

Using the dimension of K, one can find a transversal family of f_x, so that we can treat them like a transversal family of projections.

Recall that for $s \in (0,1)$.

$$\mathcal{E}_s(\mu) \sim \int_{S^{d-1}} \mathcal{E}_s((\mathrm{proj}_\theta)_\# \mu)\, d\theta.$$

and for a transversal family $\{\Phi_t\}$

$$\mathcal{E}_s(\mu) \sim \int \mathcal{E}_s((\Phi_t)_\# \mu)\, dt.$$

Proposition of Orponen:

For m large enough, for $s \in (0,1)$

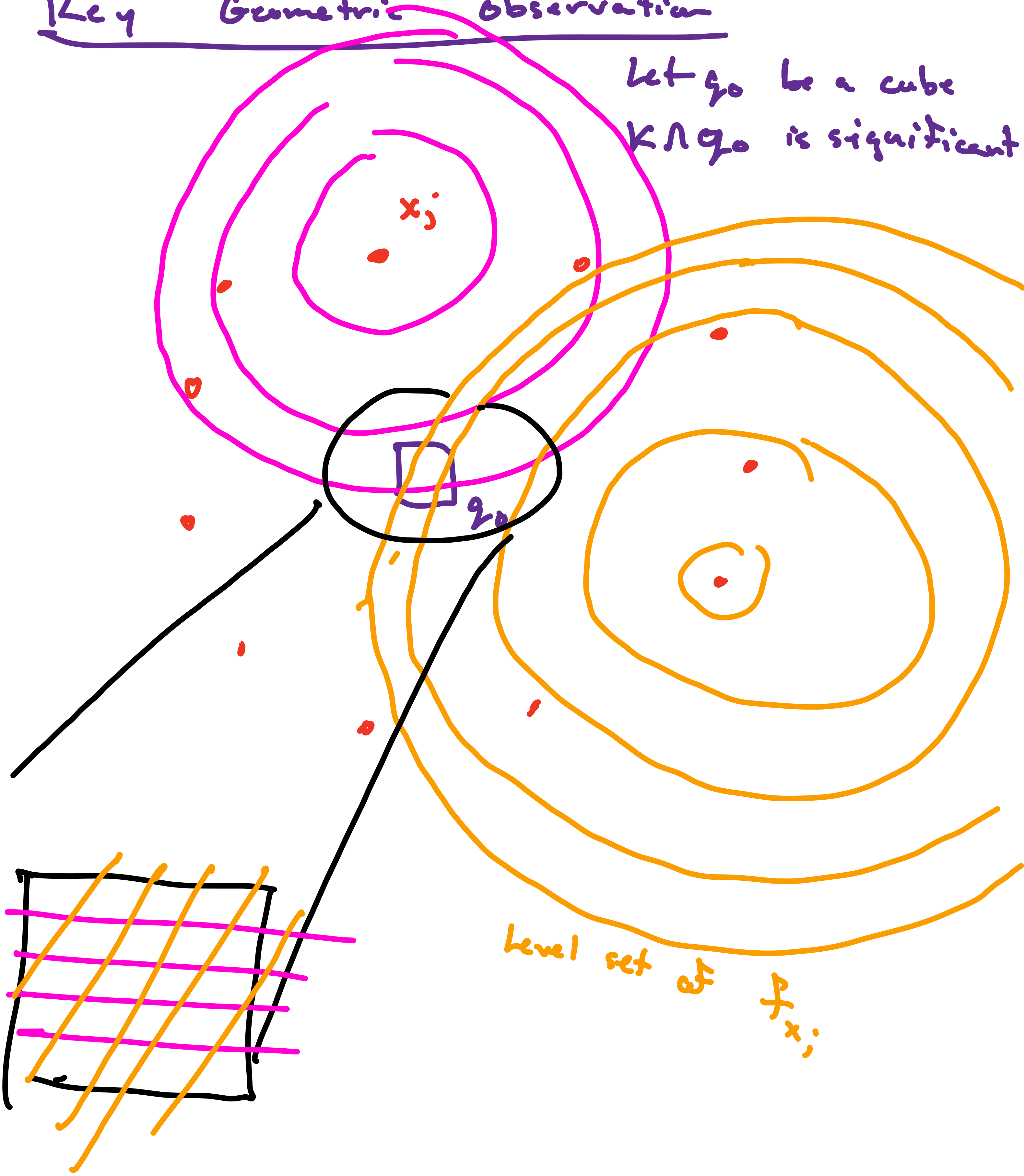
$$\int_{S^1} H_m((\mathrm{proj}_\theta)_\# \mu)\, d\theta \gtrsim s$$

$$\Rightarrow \sum_{v_\ell \in S^1} p_\ell\, H_m((\mathrm{proj}_{v_\ell})_\# \mu)) \gtrsim s - 2^{(s-1)m}.$$

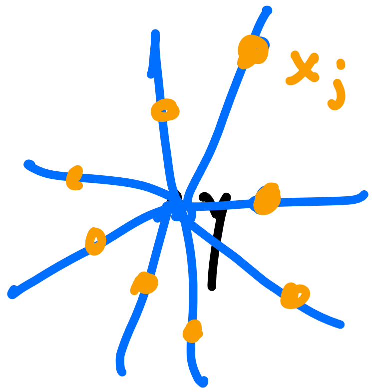
Key Geometric observation

Let q_0 be a cube

$K \cap q_0$ is significant



Since K is α-Ahlfors regular for $\alpha > 1$, for any $y \in K$ $\exists$ "points in every direction" around y.



Even distribution of directions

$\longrightarrow$ transversality.

$\Rightarrow$

$$\int_{S^1} H_m\big((F_y)_\# \mu\big) \gtrsim s$$

$$\Rightarrow \sum_{\frac{x_j}{\|x_j\|} \in S^1} p_j \cdot H_m\big((F_{x_j})_\# \mu\big) \gtrsim s - 2^{(s-1)m}.$$

Now to estimate $H_n(\ j_{\theta})_{\#}\mu)$ for arbitrarily high n, we use Hochman's local to global

$$\sum_{\frac{x_j}{\|x_j\|} \in S^2} p_j \, H_n((f_{x_j})_{\#}\mu))$$

$$\sim \sum_{\frac{x_j}{\|x_j\|} \in S^2} p_j \, \mathbb{E}_{k \leq \frac{n}{m}} \left[H[(f_{x_j})_{\#}\mu, D_{(k+1)m} | D_{km}] \right]$$

$$\sim \mathbb{E}_k \left[\sum_{\frac{x_j}{\|x_j\|} \in S^2} p_j \, H[(f_{x_j})_{\#}\mu, D_{(k+1)m} | D_{km}] \right].$$

$$\sim \mathbb{E}_{k \leq \frac{n}{m}} \left[\sum_{Q} \sum_{\frac{x_j}{\|x_j\|} \in S^2} H_m[(f_{x_j})_{\#}\mu^{Q}] \right]$$

$$\gtrsim s$$

$\Rightarrow \ H_n((p_{x_j})_{\#}\mu) > s$ for all $s \in (0,1)$. $\square$.