Fourier Decay

Lemma (Shmerkin)

Let μ, ν be compactly supported Borel measures on $\mathbb{R}$, suppose $\exists\, \alpha > 0$ s.t.

$$|\hat{\nu}(\xi)| \leq C|\xi|^{-\alpha}$$

i.) Suppose that $D_2(\mu) = 1$, then

$$\mu * \nu \ll \mathcal{L}^1 \text{ and } \frac{d(\mu*\nu)}{d\mathcal{L}^1} \in L^2$$

ii.) Suppose that

$$\underline{\dim}_H(\mu) = \inf\{\dim_H(A) \mid \mu(A) > 0\} = 1.$$

Then $\mu * \nu \ll \mathcal{L}^1$

Pf:

(i) $D_2(\mu) = 1$

$\Rightarrow \mathcal{E}_\beta(\mu) < \infty$ for all $\beta < 1$.

In particular, $\mathcal{E}_{1-2\alpha}(\mu) < \infty$.

$$\Rightarrow \quad \langle |x|^{-1+2\alpha}, \mu * \mu \rangle < \infty$$

$$\Rightarrow \quad \langle |\xi|^{-2\alpha}, |\hat{\mu}|^2 \rangle$$

$$\Rightarrow \quad \int |\xi|^{-2\alpha} |\hat{\mu}|(\xi)\, d\xi < \infty.$$

$\Rightarrow$

$$\int |\widehat{\mu * \nu}(\xi)|^2 d\xi$$

$$= \int |\hat{\mu}(\xi)|^2 |\hat{\nu}(\xi)|^2 \, d\xi$$

$$\leq \sup_{\xi} |\xi|^{2\alpha} |\hat{\nu}(\xi)|^2 \cdot \int |\xi|^{2\alpha} |\hat{\mu}(\xi)|^2 d\xi$$

$$\leq \left(\sup_{\xi} |\xi|^{2\alpha} |\hat{\nu}(\xi)|^2\right) \cdot \mathcal{E}_{1-2\alpha}(\mu).$$

iii) $\underline{\dim}_H(\mu) = 1$

$$\Rightarrow \quad \liminf_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} \geq 1 \quad \text{for } \mu\text{-a.e. } x$$

Let $\varepsilon > 0$

Egorov's thm $\Rightarrow$ $\exists A_\varepsilon$ s.t. $\mu(A_\varepsilon) > 1-\varepsilon$

s.t. $\mu_\varepsilon := \mu(A_\varepsilon)^{-1} \cdot \mu|_{A_\varepsilon}$ satisfies

$$\mu_\varepsilon(B(x,r)) \leq C_\varepsilon\, r^{1-\alpha} \qquad \forall x \in A_\varepsilon$$

$\Rightarrow \quad \dim_e(\mu_\xi) \geq 1-\alpha$

$\Rightarrow \quad \mathcal{E}_{1-2\alpha} < \infty.$ (since $\alpha > 0$).

$\Rightarrow \quad \int |\widehat{\mu * \nu}|^2 \leq \left(\sup_\xi |\xi|^{2\alpha} |\hat{\nu}(\xi)|^2\right) \cdot \mathcal{E}_{1-2\alpha}(\mu).$

$\Rightarrow \quad \frac{d(\mu * \nu)}{d\mathcal{L}^1} \in L^2 \Rightarrow \mu * \nu \ll \mathcal{L}^1$

$\square$.

Final Tool due to Erdös and Kahane.

For $s > 0$, define

$$\mathcal{F}(s) := \{ \nu \in \mathcal{M}(\mathbb{R}) \mid |\hat{\nu}(\xi)| = O(|\xi|^{-s}),\ |\xi| \to \infty \}$$

<u>Thm:</u>

$$\dim_H(\{ \lambda \in (0,1) \mid \nu_\lambda \notin \bigcup_{s>0} \mathcal{F}(s) \}) = 0$$

<u>Proof Idea</u>:

Recall that $\quad \hat{\nu}_\lambda(\xi) = \prod_{n=0}^{\infty} \cos(2\pi \lambda^n \xi)$

Let $\theta = \frac{1}{\lambda}$, $\exists N$, $t \in [1, \theta)$ $\xi = \theta^N t$.

then

$$\hat{\nu}_\lambda(\xi) = \hat{\nu}_\lambda(\theta^N t) = \left(\prod_{n=1}^{N} \cos(2\pi\, \theta^n t)\right) \cdot \hat{\nu}_\lambda(t).$$

If

$\mathrm{dist}(\theta^n t, \mathbb{Z}) > \delta > 0$ $\forall n$, then

$$\left|\prod_{n=1}^{N} \cos(2\pi\, \theta^n t)\right| \lesssim c^N \quad \text{for} \quad c \in (0,1)$$
$$\lesssim (\theta^N)^{-\beta} \lesssim |\xi|^{-\beta}$$

Therefore, if $\nu_\lambda \notin \bigcup_{s>0} \mathcal{F}(s)$, then

$$\mathrm{dist}(\theta^n t, \mathbb{Z}) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

$\Rightarrow$ $\exists$ $\{K_n\} \subset \mathbb{Z}$, $\varepsilon_n \in (-\frac{1}{2}, \frac{1}{2})$ s.t.

$\varepsilon_n \to 0$ and $\theta^n t = K_n + \varepsilon_n$.

$\Rightarrow$ $[t\theta^n]$ = fractional part of $t\theta^n \xrightarrow{n\to\infty} 0$

Claim:

$\{\theta > 1 \mid \exists t \in [1, \theta)$ s.t. $[t\theta^n] \to 0\}$
is countable

Proof of claim

Again let $t\theta^n = K_n + \varepsilon_n$. $\varepsilon_n \to 0$.

$K_n \in \mathbb{Z}$.

Now

$$\left|K_{n+2} - \frac{K_{n+1}^2}{K_n}\right| = K_{n+1}\left|\frac{K_{n+2}}{K_{n+1}} - \frac{K_{n+1}}{K_n}\right|$$

$$\leq K_{n+1}\left(\left|\theta - \frac{K_{n+2}}{K_{n+1}}\right| + \left|\theta - \frac{K_{n+1}}{K_n}\right|\right)$$

$$= K_{n+1}\left(\left|\frac{\theta K_{n+1} - K_{n+2}}{K_{n+1}}\right| + \left|\frac{\theta K_n - K_{n+1}}{K_n}\right|\right)$$

$$= K_{n+1}\left(\left|\frac{\theta(t\theta^{n+1} - \varepsilon_{n+1}) - (t\theta^{n+2} - \varepsilon_{n+2})}{K_{n+1}}\right| + \left|\frac{\theta(t\theta^n - \varepsilon_n) - (t\theta^{n+1} - \varepsilon_{n+1})}{K_n}\right|\right)$$

$$= K_{n+1}\left(\frac{|\varepsilon_{n+2} - \theta\varepsilon_{n+1}|}{K_{n+1}} + \frac{|\varepsilon_{n+1} - \theta\varepsilon_n|}{K_n}\right).$$

$$= |\varepsilon_{n+2} - \theta\varepsilon_{n+1}| + \frac{K_{n+1}}{K_n}|\varepsilon_{n+1} - \theta\varepsilon_n| \to 0$$

$n \to \infty$.

$\Rightarrow$ Eventually, K_{n+2} is uniquely determined as the nearest integer to $\dfrac{K_{n+1}^2}{K_n}$. There are only countably many sequences of this form.

$\square$.

Proof of Shmerkin's Thm

Recall:

Want to show that

$$\dim_{H}\left(\{\lambda \in (\tfrac{1}{2},1) \mid \nu_\lambda \not\ll \mathcal{L}^1\}\right) = 0.$$

First note that

$$\text{sim-dim}(\nu_\lambda) = \frac{\log(2)}{\log(\frac{1}{\lambda})} > 1.$$

since $\lambda > \frac{1}{2}$.

Let K be large enough so that

$$\left(1 - \frac{1}{K}\right)\frac{\log(2)}{\log(\frac{1}{\lambda})} > 1.$$

Recall that

$$\nu_\lambda = \mathop{\ast}_{n=1}^{\infty} \frac{1}{2}\left(\delta_{\lambda^n} + \delta_{-\lambda^n}\right).$$

$$= \left(\mathop{\ast}_{m=1}^{\infty} \frac{1}{2}\left(\delta_{\lambda^{Km}} + \delta_{-\lambda^{Km}}\right)\right) \ast \left(\mathop{\ast}_{\substack{n=1 \\ K\nmid n}}^{\infty} \frac{1}{2}\left(\delta_{\lambda^n} + \delta_{-\lambda^n}\right)\right)$$

$$= \nu_{\lambda^K} \ast \left(\mathop{\ast}_{\substack{n=1 \\ K\nmid n}}^{\infty} \frac{1}{2}\left(\delta_{\lambda^n} + \delta_{-\lambda^n}\right)\right).$$

$$=: \nu_{\lambda^k} * \nu_\lambda^k$$

ν_λ^k is generated by considering the IFS

$$\Phi^{\lambda,k} = \left(\hat{\phi}_{\underline{i}} \right)_{\underline{i} \in \Sigma_{k-1}}$$

where $\hat{\phi}_{\underline{i}} = \phi^\lambda \circ \phi^\lambda_{i_2} \circ \phi^\lambda_{i_3} \circ \cdots \circ \phi^\lambda_{i_{k-1}}$,

$$\phi^\lambda(x) = \lambda x$$

and $p_{\underline{i}} = \left(\frac{1}{2}\right)^{k-1} \quad \forall\, \underline{i} \in \Sigma_{k-1}$.

Then

$$\text{sim-dim}\left(\Phi^{\lambda,k}, p_{\underline{i}}\right) = \frac{h_p}{\chi_\lambda}$$

$$= \frac{\log\left(2^{-(k-1)}\right)}{\log\left(\lambda^k\right)}$$

$$= \left(1 - \frac{1}{k}\right) \frac{\log(2)}{\log(1/\lambda)} > 1.$$

$$= \text{sim-dim}\left(\nu_\lambda^k\right).$$

Now

1 Let

$$E_1 := \{\lambda \in (\tfrac{1}{2},1) \mid \dim_{\mathcal{H}}(\nu_\lambda^K) < \text{sim-dim}(\nu_\lambda^K)\}$$

Hochman $\Rightarrow$

$$\dim_{\mathcal{H}}(E_1)$$

and

2 Let $E_2 := \{\lambda \in (\tfrac{1}{2},1) \mid \nu_{\lambda K} \notin \bigcup_{s>0} \mathcal{F}(s)\}$.

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$$\Rightarrow \quad \dim_{\mathcal{H}}(E_2) = 0.$$

Assuming $\lambda \in (\tfrac{1}{2},1) \setminus (E_1 \cup E_2)$, Shmerkin's Lemma implies

$\nu_\lambda = \nu_{\lambda u} * \nu_\lambda^K$ has Radon-Nikodym derivative in L^2

($\exists\, s(\lambda)$ s.t. $|\widehat{\nu}_{\lambda u}(\xi)| = O(|\xi|^{-s})$

$\dim_{\mathcal{H}}(\nu_\lambda^K) = \underline{\dim}_{\mathcal{H}}(\nu_\lambda^K)$ since ν_λ^K is exact dimensional, then $\dim_{\mathcal{H}}(\nu_\lambda^K)$

$$\geq \min\left(1, \left(1-\tfrac{1}{n}\right)\frac{\log(2)}{\log(\frac{1}{\lambda})}\right),$$)