Application to Bernoulli Convolutions

Let $\lambda \in (0,1)$. Consider the iterated function system

$$\mathcal{S}^\lambda := \left(\phi_1^\lambda(x) = \lambda x - 1, \ \phi_2^\lambda(x) = \lambda x + 1 \right).$$

For each $\lambda \in [1/2, 1)$, define

$$I^\lambda := \left[\frac{-1}{1-\lambda}, \frac{1}{1-\lambda} \right].$$

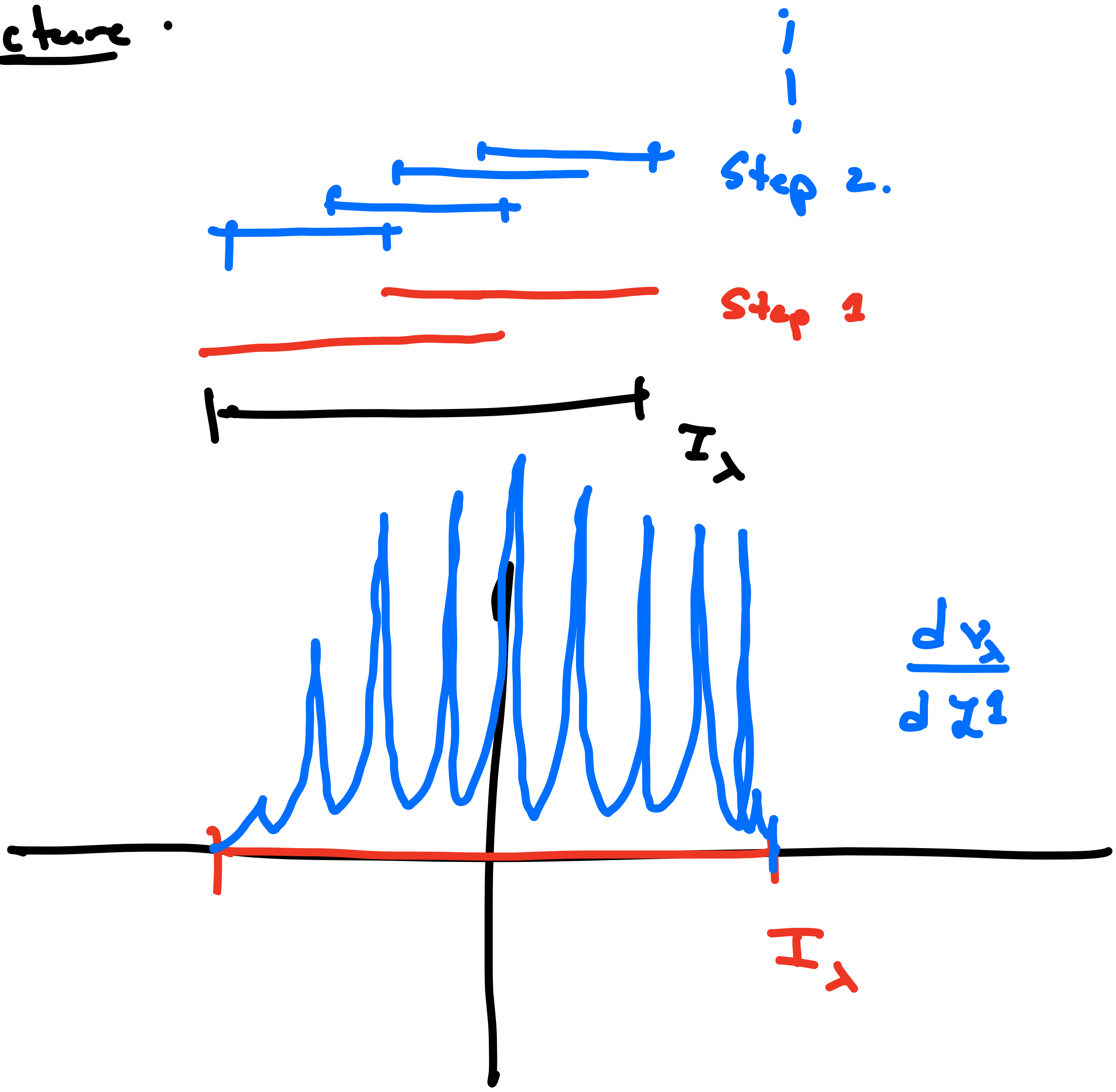
For $\lambda \in [\frac{1}{2}, 1)$, I^λ is the attractor of $\mathcal{S}^\lambda$.

Let $\mu = (\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ be the natural measure on the symbolic space $\Sigma_1 = \{1, 2\}^{\mathbb{N}}$.

Let $\Pi_\lambda : \Sigma_1 \to I_\lambda$ be the natural projection and let

$$\nu_\lambda := (\Pi_\lambda)_* \mu.$$

Picture:



Problem:

For which $\lambda \in [\frac{1}{2}, 1)$ is the measure ν_λ absolutely continuous w.r.t. Lebesgue measure?

A: Open Problem

- For $\lambda = 1/2$, ζ^λ satisfies the open set condition ✓.

<u>Motivation</u> (Erdös '30s)

Suppose we have the infinite random sum

$$Y_\lambda = \sum_{n=0}^{\infty} \pm \lambda^n$$

where $\pm$ are chosen independently with even probability. Then

$$\nu_\lambda(A) = \mathbb{P}\{Y_\lambda \in A\}$$

so ν_λ is the distribution of Y_λ.

Also, note that

$$\lim_{n\to\infty} \left(\tfrac{1}{2}(\delta_\lambda + \delta_{-\lambda})\right) * \left(\tfrac{1}{2}(\delta_{\lambda^2} + \delta_{-\lambda^2})\right) * \cdots * \left(\tfrac{1}{2}(\delta_{\lambda^n} + \delta_{-\lambda^n})\right) = \nu_\lambda.$$

$\Rightarrow$ ν_λ is called the <u>Bernoulli convolution</u>

History

- Erdős (1939), If $1/\lambda$ is a Pisot number then $\nu_\lambda \perp \mathcal{L}^1$

(ξ is a Pisot number if it is the root of an integer-coefficient polynomial with leading coefficient 1 and all of its conjugates have modulus strictly less than 1.)

- Erdős (1940), $\exists t<1$ s.t. $\nu_\lambda \ll \mathcal{L}^1$ for a.e. $\lambda \in (t,1)$

- Solomyak (1995) $\nu_\lambda \ll \mathcal{L}^1$ for a.e. $\lambda \in (\frac{1}{2},1)$.

- Hochman, Shmerkin, Varjú (More).

Thm:

For a.e. $\lambda \in (\frac{1}{2}, 1)$

$$\frac{d\nu_\lambda}{d\mathcal{L}^1} \in L^2(\mathbb{R})$$

In order to prove this, we would like to establish transversality for the family of natural projections, Π_λ.

Proof of Thm:

$\dim_H(\mu) = D_2(\mu) = 1.$

By the proof of the transversality thm, it suffices to prove that conditions 1 and 2 hold with $\alpha(\lambda) < 1$

Show condition 1

Let $\underline{i}, \underline{j} \in \Sigma'$ with $|\underline{i} \wedge \underline{j}| = n$.

Then
$$\Pi_\lambda(\underline{i}) = \sum \pm \lambda^k, \quad \Pi_\lambda(\underline{j}) = \sum \pm \lambda^k$$

and
$$\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j}) = \sum_{k=n+1}^{\infty} a_k \lambda^k$$

$\Rightarrow$
$$|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})| \lesssim \lambda^n = \left(2^{-n}\right)^{\frac{\log(\lambda^{-1})}{\log(2)}}$$
$$= \left[\mathrm{dist}(\underline{i}, \underline{j})\right]^{\frac{\log(\lambda^{-1})}{\log(2)}}$$

Let $\alpha(\lambda) = \dfrac{\log(\lambda^{-1})}{\log(2)}$

then $\alpha(\lambda) < 1$ for $\lambda > 1/2$.

and
$$|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})| \lesssim \left(\mathrm{dist}(\underline{i}, \underline{j})\right)^{\alpha(\lambda)}$$

<u>Show condition 2</u>

Again

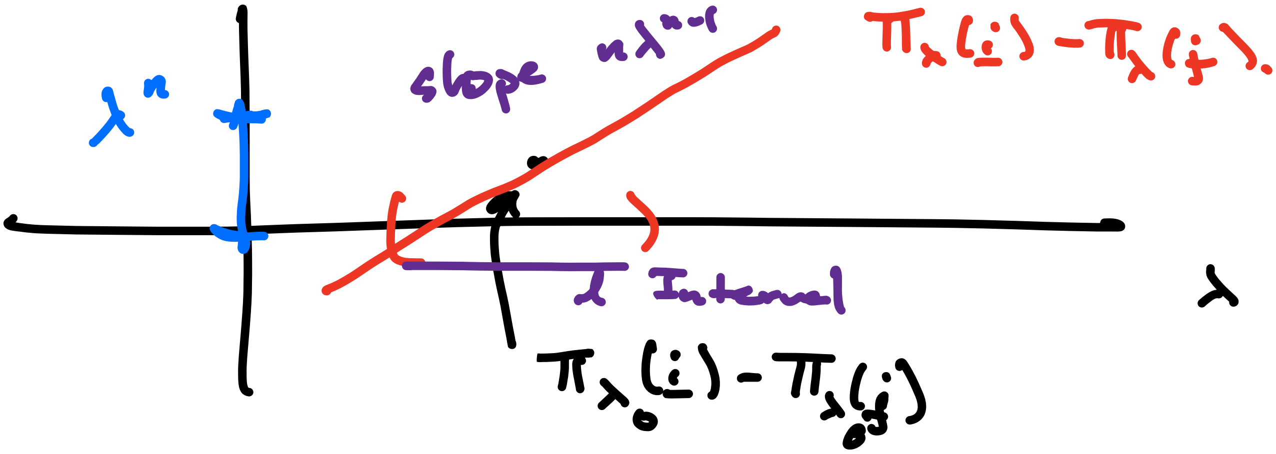
$$\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j}) = \sum_{k=n+1}^{\infty} a_k \lambda^k$$

Then

$$\frac{1}{100}\lambda^n \le |\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})| < \lambda^n.$$

$$\Rightarrow \quad \Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j}) \approx \lambda^n + \sum_{k=n+1}^{\infty} a_k \lambda.$$

$$\frac{d}{d\lambda}\left(\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})\right) \sim n\lambda^{n-1}$$



$$\frac{\lambda^n}{l} \gtrsim \lambda^{n-1} \Rightarrow l \lesssim \lambda^n \lambda^{-n+1}$$

Thus

$$\eta\left(\left\{\lambda \in [0,1] \;\middle|\; |\pi_\lambda(\underline{i}) - \pi_\lambda(\underline{j})| < r\right\}\right)$$

$$\leq \frac{r}{\lambda^n} \sim \frac{r}{\mathrm{dist}(\underline{i},\underline{j})^{\alpha(\lambda)}}.$$

Thus, by the proof of transversality thm,

$$\frac{d\nu_\lambda}{d\mathcal{L}^1} \in L^2(\mathbb{R}) \quad \text{for a.e. } \lambda.$$

$\square$.

A more careful computation of the exceptional set of Bernoulli convolutions

Thm (Peres–Schlag '00)

For every $\lambda_0 > \frac{1}{2}$, $\exists\, \varepsilon(\lambda_0) > 0$ such that

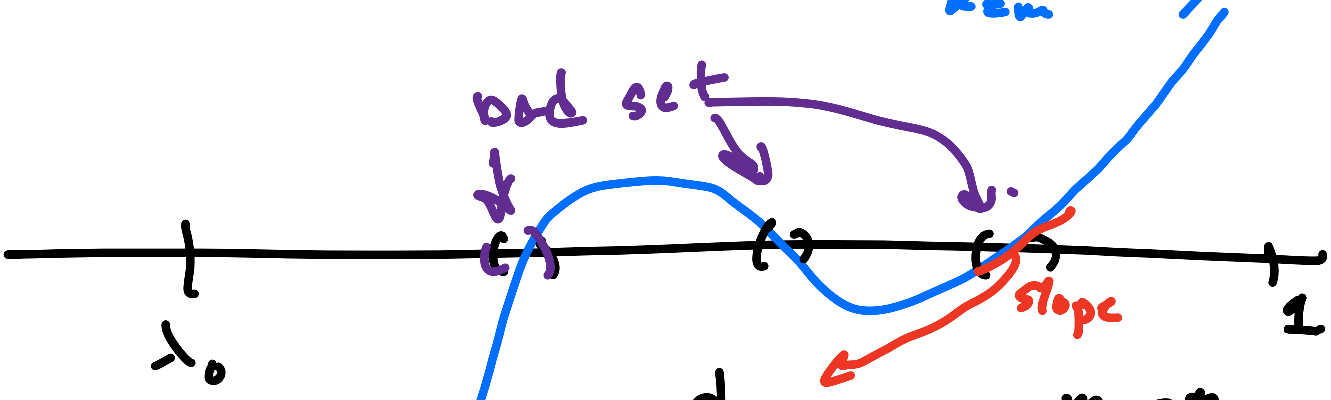
$$\dim_{\mathcal{H}}\left(\left\{ \lambda \in (\lambda_0, 1) \;\middle|\; \nu_\lambda \text{ does not have } L^2 \text{ density} \right\}\right)$$

$$\leq 1 - \varepsilon(\lambda_0).$$

Idea: For $n \in \mathbb{Z}^+$,

$r > 0$

$$g(\lambda) = \sum_{k=m}^{\infty} a_k \lambda^k \Big/ 2^{-m}$$



$$\frac{d}{d\lambda} g(\lambda) \gtrsim \lambda^m 2^m$$

$$= 2^{m\left[1 - \frac{\log\left(\frac{1}{\lambda}\right)}{\log(2)}\right]}$$

The larger the λ_0, the larger the lower bound on the slope.
$\Rightarrow$ Cantor set of "bad" λ has smaller dimension than 1 and this dimension decreases as $\lambda_0 \to 1$.

Better Estimates of Shmerkin

(simple version)

Thm: $\exists\ E \subset (0,1)$ s.t. $\dim_H(E) = 0$ and if $\lambda \in (\frac{1}{2}, 1) \setminus E$, then

$$\nu_\lambda \ll \mathcal{L}^1 \quad \text{and} \quad \frac{d\nu_\lambda}{d\mathcal{L}^1} \in L^2(\mathbb{R}).$$

Some tools.

<u>Cor:</u> (Corollary to Hochman's thm)
Let $I \subset \mathbb{R}$ be a close interval. Let $r_i : I \to (-1,1)\setminus\{0\}$ and $d_i : I \to \mathbb{R}$ be real-analytic.

Consider the parametric family of IFS's:

$$\Phi_t = \left(\phi_{i,t}(x) = r_i(t)x + d_i(t) \right)_{i=1}^{m}$$

Let π_t be the associated natural projection. Suppose that

$$\forall \underline{i}, \underline{j} \in \Sigma_1', \quad \pi_t(\underline{i}) \equiv \pi_t(\underline{j})$$

$$\iff \underline{i} = \underline{j}.$$

For a probability vector $p = (p_1, \dots, p_m)$ let ν_t^p be the corresponding self-similar measure. Then the set

$$\{ t \in I \mid \exists p \text{ s.t. } \dim_H(\nu_t^p) < \min(\text{sim-dim}(\Phi_t), 1) \}$$

has Hausdorff and Packing dimension 0.