Idea:

Part (i) of the projection thm for sets and parts (i) and (iii) of the projection thm for sets are akin to the Marstrand thm we proved before:

$$\mathcal{E}_s(\mu) \gtrsim \int_V \mathcal{E}_s(P_{V\#}\mu)\, dV.$$

$\Rightarrow$ for a.e. V and $\forall \varepsilon > 0$,

$$\dim_H(P_{V\#}\mu) \geq \dim_H(\mu) - \varepsilon.$$

$\Rightarrow$

$$\dim_H(P_{V\#}\mu) \geq \dim_H(\mu) \text{ for a.e. } V.$$

Parts (ii) of the projection thm for sets (ii) and (iv) of the projection thm for measures is different and handles the case in which $\dim_H(V) < \dim_H(\mu)$ in which case

$$\dim_H(P_{V\#}\mu) \geq \dim_H(\mu) \text{ is not possible.}$$

We can still use s-energies for this case.

Suppose $m = \dim_{\mathcal{H}}(V) < \dim_{\mathcal{H}}(\mu)$.

We can think of $P_{V\#}\mu$ as a measure on $\mathbb{R}^m$, for each $V \in G(d,m)$.

We still get

$$\infty > \mathcal{E}_s(\mu) \gtrsim_{\mu} \int_{G(d,m)} \mathcal{E}_s(P_{V\#}\mu)\, \sigma(dV).$$

Some Harmonic Analysis.

Let $\varphi_s(x) := \|x\|^{-s}$, $\varphi_s : \mathbb{R}^d \to \mathbb{R}$, then

$$\hat{\varphi}_s(\xi) = \varphi_{d-s}(\xi) = \|\xi\|^{s-d}$$

Thus

$$\mathcal{E}_s(\nu) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \|x-y\|^{-s}\, \nu(dx)\, \nu(dy)$$

$$= \langle \varphi_s, \nu * \nu \rangle = \langle \hat{\varphi}_s, \hat{\nu}\, \bar{\hat{\nu}} \rangle$$

$$= \int |\xi|^{s-m} |\hat{\nu}(\xi)|^2\, d\xi.$$

$$= \int_{\mathbb{R}^m} \left(|\xi|^{\frac{s-m}{2}} |\hat{\nu}(\xi)| \right)^2 d\xi.$$

$$= \|\nu\|^2_{\dot{H}^{\frac{s-m}{2}}}.$$

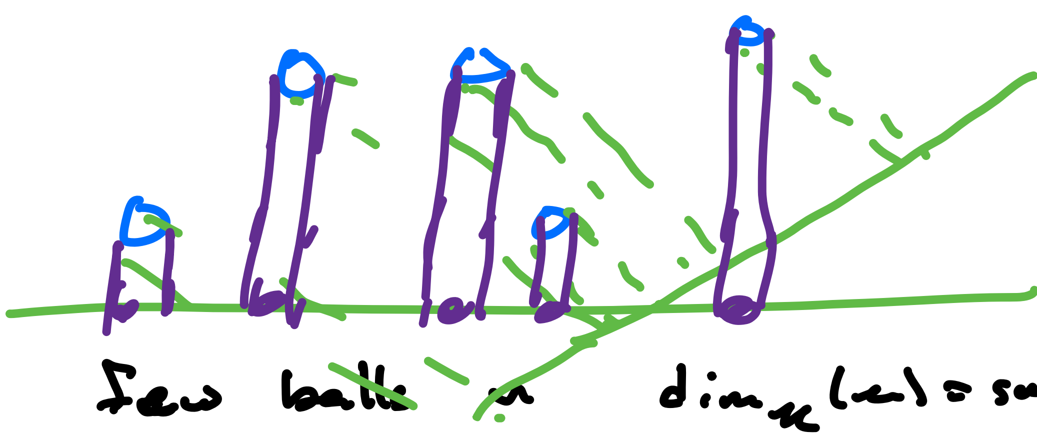
If ν is finite,

$$\left\| \frac{d\nu}{d\mathcal{L}^m} \right\|_{L^2(\mathbb{R}^m)} \leq C + \mathcal{E}_s(\nu) \quad \text{if } s \geq m.$$

$$\Rightarrow \quad \nu \ll \mathcal{L}^m.$$

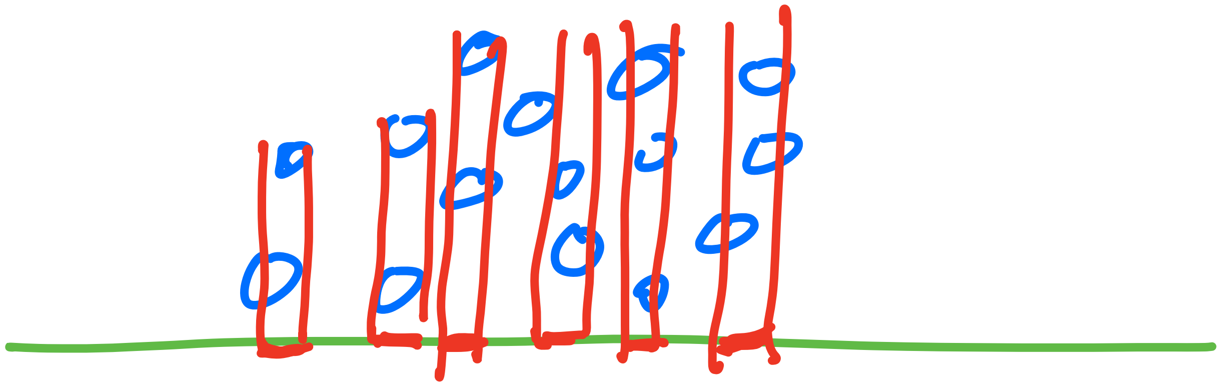
Geometric Idea

Consider N r-balls in the plane



Very few balls on $\dim_{\mathcal{H}}(\ldots) = \text{small}$.

Most projections will see N distinct balls.



Very many balls & $\dim_\mu(x) = $ big.

Most projections will see lots of overlap but an "equal" amount of overlap $\Rightarrow$ regularity.

Proof of part (iii) of projection thm for measures

Goal: Show that

$$D_2(\nu_\lambda) = \min\left(d, \frac{D_2(\mu)}{\alpha(\lambda)}\right) \text{ for } \eta\text{-a.e. } \lambda \in U.$$

$$D_2(\nu_\lambda) \geq \min\left(d, \frac{D_2(\mu)}{\alpha(\lambda)}\right)$$

(Note: since ν_λ is not trivial, $D_2(\nu_\lambda) \leq d$.)

we observe, as before, that for $t \geq 0$

$$\mathcal{E}_t(\nu_\lambda) = \int_X \int_X \| \Phi_\lambda(x) - \Phi_\lambda(y) \|^{-t} \, d\mu(x) \, d\mu(y)$$

$\Longrightarrow$

$$\int_U \mathcal{E}_t(\nu_\lambda) \, d\eta = \iint \int_U \| \Phi_\lambda(x) - \Phi_\lambda(y) \|^{-t} \, d\eta \, d\mu(x) \, d\mu(y)$$

$$= \iint \int_0^\infty \eta\left(\{ \| \Phi_\lambda(x) - \Phi_\lambda(y) \| < r^{-\frac{1}{t}} \}\right) dr$$

$$\lesssim \iint \left(\int_0^{\rho(x,y)^{t\alpha(\lambda)}} 1 \right) + \int_{\rho(x,y)^{t\alpha(\lambda)}}^\infty \frac{r^{-\frac{d}{t}}}{\rho(x,y)^{d(\alpha(\lambda_0)+\varepsilon)}}$$

$$\lesssim \iint \left(\rho(x,y) \right)^{t\alpha(\lambda)} \, d\mu(x) \, d\mu(y)$$

$$= \mathcal{E}_{t\alpha(\lambda)}(\mu).$$

$$\Rightarrow \int D_2(\nu_\lambda) \geq \inf \{ s \geq 0 \mid \int \mathcal{E}_s(\nu_\lambda) = \infty \}$$

$$\geq \inf \{ s \geq 0 \mid \mathcal{E}_{s\alpha(\lambda)}(\mu) = \infty \}$$

$$= D_2(\mu) / \alpha(\lambda).$$

$$\underline{D_2(\nu_\lambda) \le \min\left(d, \frac{D_2(\mu)}{\alpha(\lambda)}\right).}$$

$$\mathcal{E}_t(\nu_\lambda) = \iint \|\Phi_\lambda(x) - \Phi_\lambda(y)\|^{-t} \mu(dx)\mu(dy)$$

$$\overset{1}{\ge} C(\lambda)^{-t} \iint \rho(x,y)^{-t\alpha(\lambda)} \mu(dx)\mu(dy)$$

$$= C(\lambda)^{-t} \mathcal{E}_{t\alpha(\lambda)}(\mu).$$

$\Rightarrow$ For all λ,

$$\frac{D_2(\mu)}{\alpha(\lambda)} \le D_2(\nu_\lambda).$$

$\square$

Proof of part (iv) of projection thm for measures

Let $\lambda_0 \in U$ satisfy $\frac{D_2(\mu)}{\alpha(\lambda_0)} > d$.

Denote $\alpha_0 = \alpha(\lambda_0)$. Choose $\beta < D_2(\mu)$ s.t.

$$d < \frac{\beta}{\alpha_0} < \frac{D_2(\mu)}{\alpha_0}$$

Then $\mathcal{E}_\beta(\mu) < \infty$.

Fix $\varepsilon > 0$ s.t. $\beta > d(\alpha_0 + \varepsilon)$

Let $N_{\varepsilon,\lambda_0}$ be a nbhd of λ_0 s.t.

$$\eta\left(\{\lambda \in N_{\varepsilon,\lambda_0} \mid \|\Phi_\lambda(x) - \Phi_\lambda(y)\| \le r\}\right) \lesssim \frac{r^d}{\rho(x,y)^{d(\alpha_0+\varepsilon)}}$$

Let $\underline{D}(\nu_\lambda, \xi) = \liminf_{r \to 0} \frac{\nu_\lambda(B(\xi,r))}{\mathcal{L}^d(B(\xi,r))}$

<u>Note:</u> It suffices to show that $\underline{D}(\nu_\lambda, x) < \infty$ for ν_λ-a.e. $x \in \mathbb{R}^d$ in order to conclude that $\nu_\lambda \ll \mathcal{L}^d$.

Goal:
We will show that

$$J := \int_{N_{\varepsilon,\lambda_0}} \int_{\mathbb{R}^d} \underline{D}(\nu_\lambda, \xi)\, \nu_\lambda(d\xi)\, \eta(d\lambda) < \infty.$$

To prove the goal, note that

$$J = \iint_{N_{\varepsilon,\lambda_0}\ \mathbb{R}^d} \liminf_{r\to 0} \frac{\nu_\lambda(B(\xi,r))}{\mathcal{L}^d(B(\xi,r))}\, \nu_\lambda(d\xi)\, \eta(d\lambda)$$

$$\leq C_d \liminf_{r\to 0} r^{-d} \iint_{N_{\varepsilon,\lambda_0}\ \mathbb{R}^d} \nu_\lambda(B(\xi,r))\, \nu_\lambda(d\xi)\, \eta(d\lambda)$$

$$= C_d \liminf_{r\to 0} r^{-d} \iint_{N_{\varepsilon,\lambda_0}\ \mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{B(\xi,r)}(\tau)\, \nu_\lambda(d\tau)\, \nu(d\xi)\, \eta(d\lambda)$$

$$= C_d \liminf_{r\to 0} r^{-d} \iiint \chi_{\{\|\Phi_\lambda(x)-\Phi_\lambda(y)\|<r\}}\, \mu(dx)\, \mu(dy)\, \eta(d\lambda)$$

$$= C_d \liminf_{r\to 0} r^{-d} \iint_{X\ X} \eta(\{\lambda \in N_{\varepsilon,\lambda_0} \mid \|\Phi_\lambda(x)-\Phi_\lambda(y)\|<r\})\, \mu(dx)\, \mu(dy)$$

$$\lesssim \liminf_{r\to 0} r^{-d} \iint r^d \left(\rho(x,y)\right)^{d(\alpha_0+\varepsilon)}\, \mu(dx)\, \mu(dy)$$

$$\leq \mathcal{E}_\beta(\mu) \qquad \left(\text{since } \beta > d(\alpha_0+\varepsilon)\right).$$

$\Rightarrow \quad \nu_\lambda \ll \mathcal{L}^d$ for a.e. $\lambda \in N_{\varepsilon,\lambda_0}$.

Moreover, for such λ

$$\int \underline{D}(\nu_\lambda, \xi)\, \nu_\lambda(d\xi)$$

$$= \int \frac{d\nu_\lambda}{d\mathcal{L}^d}(\xi)\, \nu_\lambda(d\xi)$$

$$= \int \left(\frac{d\nu_\lambda}{d\mathcal{L}^d}(\xi)\right)^2 d\xi = \left\| \frac{d\nu_\lambda}{d\mathcal{L}^d} \right\|_{L^2}^2 < \infty.$$