Projections

Now is a good time to discuss projections more in-depth. We will start with a very general notion of a transversal family of generalized projections, then prove a general theorem. This will give us a good understanding of the main ideas. We will keep these in mind when we discuss specific examples.

Riesz energy and Correlation Dimension in metric-measure spaces

Let μ be a Borel probability measure on a metric space (X, ρ)

Generalized s-energy

For $s > 0$

$$\mathcal{E}_s(\mu) := \int_X \int_X (\rho(x,y))^{-s} \, d\mu(x) \, d\mu(y)$$

Recall:

Lemma: If μ is s-Frostman, then

$$\mathcal{E}_t(\mu) < \infty \quad \text{for } t < s.$$

Lemma: If $\mathcal{E}_s(\mu) < \infty$, then $\dim_{\mathcal{H}}(\mu) \geq s$

For $\varepsilon > 0$, define

$$C_2(\mu, \varepsilon) := \int_X \mu(B(x,r)) \, d\mu(x)$$

(**Note:** if $(X, \rho) = (\mathbb{R}^d, \|\cdot\|)$, then

$$C_2(\mu, \varepsilon) = (\mu * \mu)(B(0,r))$$
)

Def: The <u>lower correlation dimension</u> of μ is defined by

$$D_2(\mu) := \liminf_{\varepsilon \to 0} \frac{\log(C_2(\mu,\varepsilon))}{\log(\varepsilon)}$$

The <u>upper correlation dimension</u> of μ is defined by

$$\overline{D}_2(\mu) := \limsup_{\varepsilon \to 0} \frac{\log(C_2(\mu,\varepsilon))}{\log(\varepsilon)}$$

Prop:

$$\begin{aligned} D_2(\mu) &= \sup\{s \geq 0 \mid \mathcal{E}_s(\mu) < \infty\} \\ &= \inf\{s \geq 0 \mid \mathcal{E}_s(\mu) = \infty\}. \end{aligned}$$

This immediately provides the next lemma

Lem: $D_2(\mu) \leq \dim_H(\mu)$.

Def: (Transversal Family of Projections)

Let (X, ρ) be a compact metric space, let U be a separable compact metric space with Radon measure η. Suppose one has a family of maps $\{\Phi_\lambda\}_{\lambda \in U}$

$$\Phi_\lambda : X \to \mathbb{R}^d .$$

where $\{\Phi_\lambda\}$ satisfies

1) (Hölder continuity)

$$\| \Phi_\lambda(x) - \Phi_\lambda(y) \| \leq C(\lambda) \cdot \rho(x,y)^{\alpha(\lambda)}$$

$$x, y \in X , \quad \lambda \in U$$

where $C(\lambda)$, $\alpha(\lambda)$ are continuous in λ.

2) (Transversality)

For any $\lambda_0 \in U$ and $\varepsilon > 0$ $\exists$ nbhd $N_{\varepsilon, \lambda_0}$ of λ_0 such that $\forall r > 0$,

$$\eta\left(\{ \lambda \in N_{\varepsilon, \lambda_0} \mid \| \Phi_\lambda(x) - \Phi_\lambda(y) \| \leq r \} \right)$$

$$\leq C_{\lambda_0, \varepsilon} \frac{r^d}{\rho(x,y)^{d(\alpha(\lambda_0) + \varepsilon)}}$$

Basic Example.

$X \subset \mathbb{R}^2$ compact, $\rho = \|\cdot - \cdot\|$

$U = S^1$, η = Haar measure on S^1

$\Phi_\lambda(x) := \langle x, \lambda \rangle$ for $\lambda \in S^1$

Then $\alpha(\lambda) = 1$ for all $\lambda \in S^1$

$C(\lambda) = 1 \quad \forall \lambda \in S^1$

(1) $|\Phi_\lambda(x) - \Phi_\lambda(y)| \le \|\lambda\| \|x-y\| = \|x-y\|$

(2) Let $\lambda_0 \in S^1$, $\epsilon > 0$, $N_{\lambda_0, \epsilon} = B(\lambda_0, \epsilon)$

For $r \ll \|x-y\|$

$|\Phi_\lambda(x) - \Phi_\lambda(y)| < r$

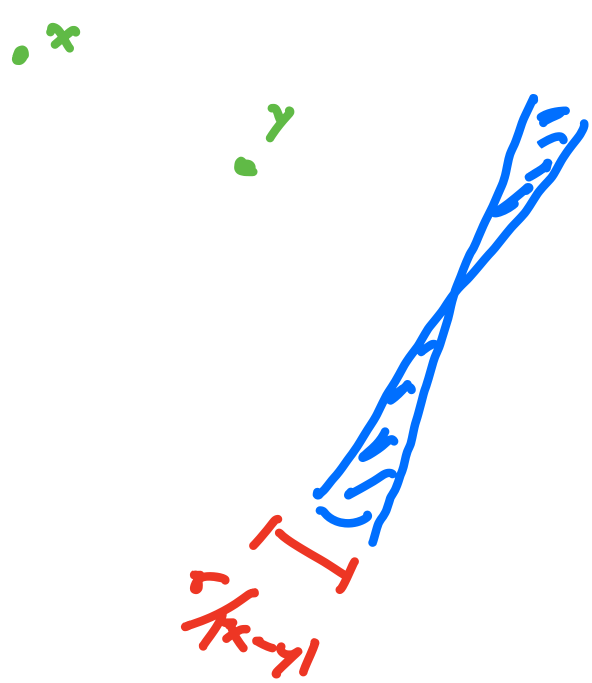
$\iff |\langle \lambda, x-y \rangle| < r$

$\iff |\langle \lambda^1 + \lambda^2, x-y \rangle| < r$

$\iff |\langle \lambda^1, x-y \rangle| < r$

$\iff \|\lambda^2\| \, |x-y| < r$

$\iff \|\lambda^2\| < r/|x-y|$



Projection Thm for sets.

Thm:

Assume (1) and (2) hold. Then

i) $\dim_{\mathcal{H}}(\Phi_\lambda(X)) = \min\left(d, \frac{\dim_{\mathcal{H}}(X)}{\alpha(\lambda)}\right)$

for η-a.e. $\lambda \in U$.

ii) $\mathcal{L}^d(\Phi_\lambda(X)) > 0$ for η-a.e. λ

satisfying $\frac{\dim_{\mathcal{H}}(X)}{\alpha(\lambda)} > d$

Projection Thm for Measures.

Assume ① and ② hold. Let μ be a Borel measure on X, and define

$$\nu_\lambda := \mu \circ \Phi_\lambda^{-1} \in \{\text{Borel measures on } \mathbb{R}^d\}.$$

Then

i.) $\dim_H(\nu_\lambda) = \min\left(d, \dfrac{\dim_H(\mu)}{\alpha(\lambda)}\right)$ for η-a.e. $\lambda \in U$

ii.) $\nu_\lambda \ll \mathcal{L}^d$ for η-a.e. $\lambda \in U$ such that

$$\frac{\underline{\dim}_H(\mu)}{\alpha(\lambda)} > d.$$

could this be improved to $\dfrac{\dim_H(\mu)}{\alpha(\lambda)} > d.$

iii.) $D_2(\nu_\lambda) = \min\left(d, \dfrac{D_2(\mu)}{\alpha(\lambda)}\right)$ for η-a.e. U

iv.) $\nu_\lambda \ll \mathcal{L}^d$ with density in $L^2(\mathbb{R}^d)$ for η-a.e. $\lambda \in U$ such that

$$\frac{D_2(\mu)}{\alpha(\lambda)} > d.$$

Idea:

Part (i) of the projection thm for sets and parts (i) and (iii) of the projection thm for sets are akin to the Marstrand thm we proved before:

$$\mathcal{E}_s(\mu) \gtrsim \int_V \mathcal{E}_s(P_{V\#}\mu)\, dV.$$

$\Rightarrow$ for a.e. V and $\forall \varepsilon>0$,

$$\dim_H(P_{V\#}\mu) \geq \dim_H(\mu)-\varepsilon.$$

$\Rightarrow$ $\dim_H(P_{V\#}\mu) \geq \dim_H(\mu)$ for a.e. V.

Parts (ii) of the projection thm for sets (ii) and (iv) of the projection thm for measures is different and handles the case in which $\dim_H(V) < \dim_H(\mu)$ in which case

$\dim_H(P_{V\#}\mu) \geq \dim_H(\mu)$ is not possible.

We can still use s-energies for this case.

Suppose $m = \dim_{\mathcal{H}}(V) < \dim_{\mathcal{H}}(\mu)$.

We can think of $P_{V\#}\mu$ as a measure on $\mathbb{R}^m$, for each $V \in G(d,m)$.

We still get

$$\infty > \mathcal{E}_s(\mu) \gtrsim \int_{G(d,m)} \mathcal{E}_s(P_{V\#}\mu)\, \sigma(dV).$$