Now that we've proven that a gap between similarity dimension implies super-exponential condensation.

Hochman's thm would imply a resolution to the exact overlap conjecture if super-exponential condensation implies exact overlap.

We have shown that this holds when the contraction ratios and translations are rational. So it stands to reason that this is possible. In fact, Hochman mentions in his paper that this may be true.

Unfortunately, it is not true. This is shown by Bárány and Käenmäki then separately by Baker. We will prove the Bárány - Käenmäki result.

Setup for Bárány – Käenmäki.

A parametrized family of maps.

Let $r \in (0, \frac{1}{3})$

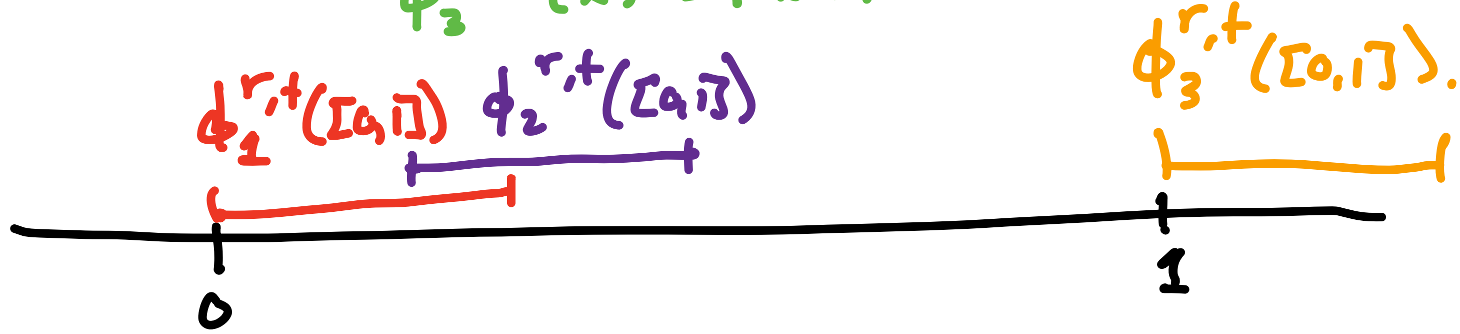
$t \in (0, \frac{r}{1-r})$

Define

$$\phi_1^{r,t}(x) = rx$$

$$\phi_2^{r,t}(x) = rx + t$$

$$\phi_3^{r,t}(x) = rx + 1$$



Let $\Phi_{r,t} := (\phi_1^{r,t}, \phi_2^{r,t}, \phi_3^{r,t})$.

and let $\Lambda_{r,t}$ = attractor of $\Phi_{r,t}$.

Notes:

- $\phi_1^{r,t}(co(\Lambda_{r,t})) \cap \phi_2^{r,t}(co(\Lambda_{r,t})) \neq \emptyset$
- $\phi_2^{r,t}(co(\Lambda_{r,t})) \cap \phi_3^{r,t}(co(\Lambda_{r,t})) = \emptyset$
- sim-dim$(\Phi_{r,t}) = \log(3)/\log(\frac{1}{r}) < 1$.

Parameter Sets

- **Exact overlapping set**

$$\mathcal{E} = \left\{ (r,t) \;\middle|\; \phi_{\underline{i}}^{r,t} = \phi_{\underline{j}}^{r,t} \text{ for } \underset{\underline{i} \neq \underline{j}}{\underline{i}, \underline{j} \in \Sigma_1^*} \right\}$$

- **Dimension Drop set**

$$\mathcal{D} = \left\{ (r,t) \;\middle|\; \dim_{\mathcal{H}}(\Delta_{r,t}) < \text{sim-dim}(\Phi_{r,t}) \right\}.$$

- **Super-Exponential Condensation Set**

$$\mathcal{C} = \left\{ (r,t) \;\middle|\; \lim_{n \to \infty} \frac{1}{n} \log(\Delta_n^{r,t}) = -\infty \right\}.$$

where

$$\Delta_n^{r,t} := \min\left\{ |\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| \;\middle|\; \underset{\underline{i} \neq \underline{j}}{\underline{i}, \underline{j} \in \Sigma_1^n} \right\}.$$

- **η-condensation set**

Given a monotone decreasing sequence of positive real numbers, $\eta = (\eta_k)_{k=1}^{\infty}$

$$\mathcal{C}_\eta := \left\{ (r,t) \;\middle|\; \Delta_k^{r,t} < \eta_k \;\; \forall k \in \mathbb{N} \right\}$$

(If $\lim_{k \to \infty} \frac{1}{k} \log(\eta_k) = -\infty$, then $\mathcal{C}_\eta \subset \mathcal{C}$).

Thm (Bárány–Käenmäki '20)

IF $\lim_{n\to\infty} \frac{1}{n} \log(\eta_n) = -\infty$

then $\mathcal{C}_\eta \setminus \mathcal{E}$ is uncountable

Tools for the proof

Average Exponential Condensation

For any IFS, $\Phi = (\phi_1, \phi_2, \ldots, \phi_m)$

$$X(\gamma) = \liminf_{n\to\infty} \frac{1}{n} \sum_{\underline{i} \in \Sigma_n} \frac{1}{m^n} \log\left(\#\left\{ \underline{j} \in \Sigma_n \,\middle|\, |\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| \le \gamma^n \right\} \right)$$

$X(\gamma)$ is essentially computing the avg Box-counting dimension of the fibers of the natural projection over all scales.

This ends up being close to the projection entropy of Feng and Hu. We can intuit that fact from the following proposition.

<u>Prop:</u> If $\Phi = (\phi_1, \phi_2, \ldots, \phi_m)$ is a homogeneous tuple of contractive similitudes acting on the real line such that r with $0 < |r| < \frac{1}{m}$ is the common contraction ratio of the maps, ϕ_i, $\Delta \subset \mathbb{R}$ is the associated self-similar set, and μ is the natural measure on X, then

$$\dim_{\kappa}(\mu) = \text{sim-dim}(\Phi) - \frac{X(\gamma)}{\log(|r|^{-1})}.$$

For all $0 < \gamma \leq |r|$.

This proposition isn't necessary for the construction, but it is a nice application of Hochman's entropy computations.

Sketch of proof of Bárány-Käenmäki thm

For $j, i \in \{1, 2, 3\}$,

let $\delta_i^j = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$

Then for $\underline{i} \in \Sigma_n$, $\underline{i} = (i_1, \ldots, i_n)$

$$\phi_{\underline{i}}^{r,t}(0) = \sum_{k=1}^{n} \left(\delta_{i_k}^3 + t\delta_{i_k}^2 \right) r^{k-1}$$

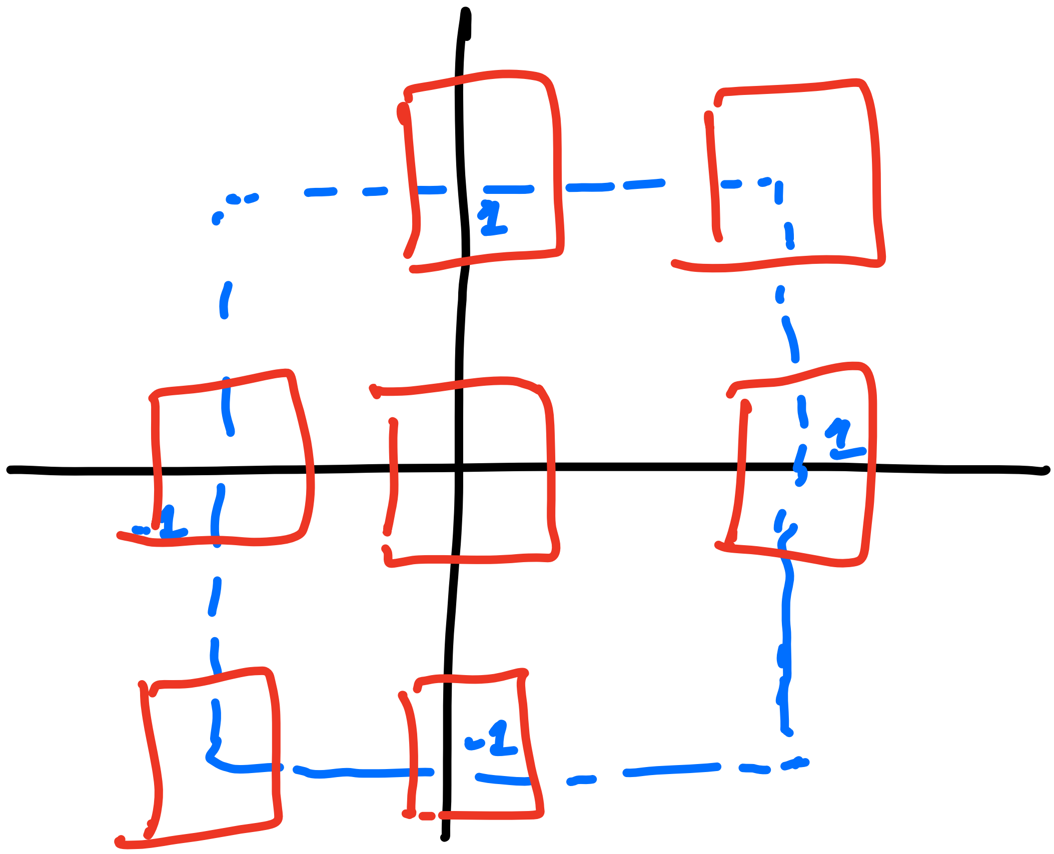
Thus,

$$\left| \phi_{\underline{i}}^{r,t}(0) - \phi_{\underline{j}}^{r,t}(0) \right| < \varepsilon$$

$$\Rightarrow \left| t - \frac{\sum_{k=1}^{n} \left(\delta_{j_k}^3 - \delta_{i_k}^3 \right) r^{k-1}}{\sum_{k=1}^{n} \left(\delta_{i_k}^2 - \delta_{j_k}^2 \right) r^{k-1}} \right| < 2\varepsilon$$

and

$$\left| t - \frac{\sum_{k=1}^{n} \left(\delta_{j_k}^3 - \delta_{i_k}^3 \right) r^{k-1}}{\sum_{k=1}^{n} \left(\delta_{i_k}^2 - \delta_{j_k}^2 \right) r^{k-1}} \right| < \varepsilon \Rightarrow \left| \phi_{\underline{i}}^{r,t}(0) - \phi_{\underline{j}}^{r,t}(0) \right| < 2\varepsilon.$$



Define an IFS

$$\Phi_r = \left(\psi^r_{(l_1,l_2)}\right)_{(l_1,l_2)\in L}$$

where $L = \{(0,0), (-1,0), (1,0), (0,-1), (0,1), (-1,-1), (1,1)\}$

and

$$\psi^r_{(l_1,l_2)} : \mathbb{R}^2 \to \mathbb{R}^2$$

and $\psi^r_{(l_1,l_2)}(x) = rx + (l_1,l_2).$

Further define the map

$$\beta(i,j) = (\delta_j^3 - \delta_i^3, \delta_i^2 - \delta_j^2)$$

for $i, j \in \{1,2,3\}$.

Then

$$\beta : \{1,2,3\} \times \{1,2,3\} \to L$$

Now for each pair, $\underline{i}, \underline{j} \in \Sigma_n$

$$\left(\sum_{k=1}^{n} (\delta_{j_k}^3 - \delta_{i_k}^3) r^{k-1}, \sum_{k=1}^{n} (\delta_{i_k}^2 - \delta_{j_k}^2) r^{k-1} \right)$$

we can write

$$(\Psi_\beta^r)(\underline{i}, \underline{j}) := \Psi^r_{(\beta(i_1,j_1))} \circ \Psi^r_{(\beta(i_2,j_2))} \circ \cdots$$

one can think of the order of compositions as reversed.

$$\cdots \circ \Psi^r_{(\beta(i_n,j_n))}(0,0)$$

$$= \left(\sum_{k=1}^{n} (\delta_{j_k}^3 - \delta_{i_k}^3) r^{k-1}, \sum_{k=1}^{n} (\delta_{i_k}^2 - \delta_{j_k}^2) r^{k-1} \right)$$

Finally, $\text{proj} : \mathbb{R}^2 - \{x_2 = 0\} \to \mathbb{R}$

be a "projection" defined by

$$\text{proj}(x_1, x_2) = \frac{x_1}{x_2}$$

For $\underline{i}, \underline{j} \in \Sigma_n$, define

$$P^r(\underline{i}, \underline{j}) := \text{proj}\left(\Psi_R^r(\underline{i}, \underline{j}) \right)$$

$$P^r : \Sigma_1^* \times \Sigma_1^* \to \mathbb{R}.$$

P^r is not necessarily defined on all of $\Sigma_1^* \times \Sigma_1^*$, but we can make some small changes by fixing some small amount of letters of each word.

P_r extends continuously to a function

$$P_r : \Sigma_1 \times \Sigma_1 \to \mathbb{R}.$$

So now $\{P_r\}_r$ is a family of "projections"

Now it suffices to find uncountably many t s.t $\exists\ \underline{i}, \underline{j} \in \Sigma_1'$

$$P^r(\underline{i}, \underline{j}) = t \quad \text{and more importantly.}$$

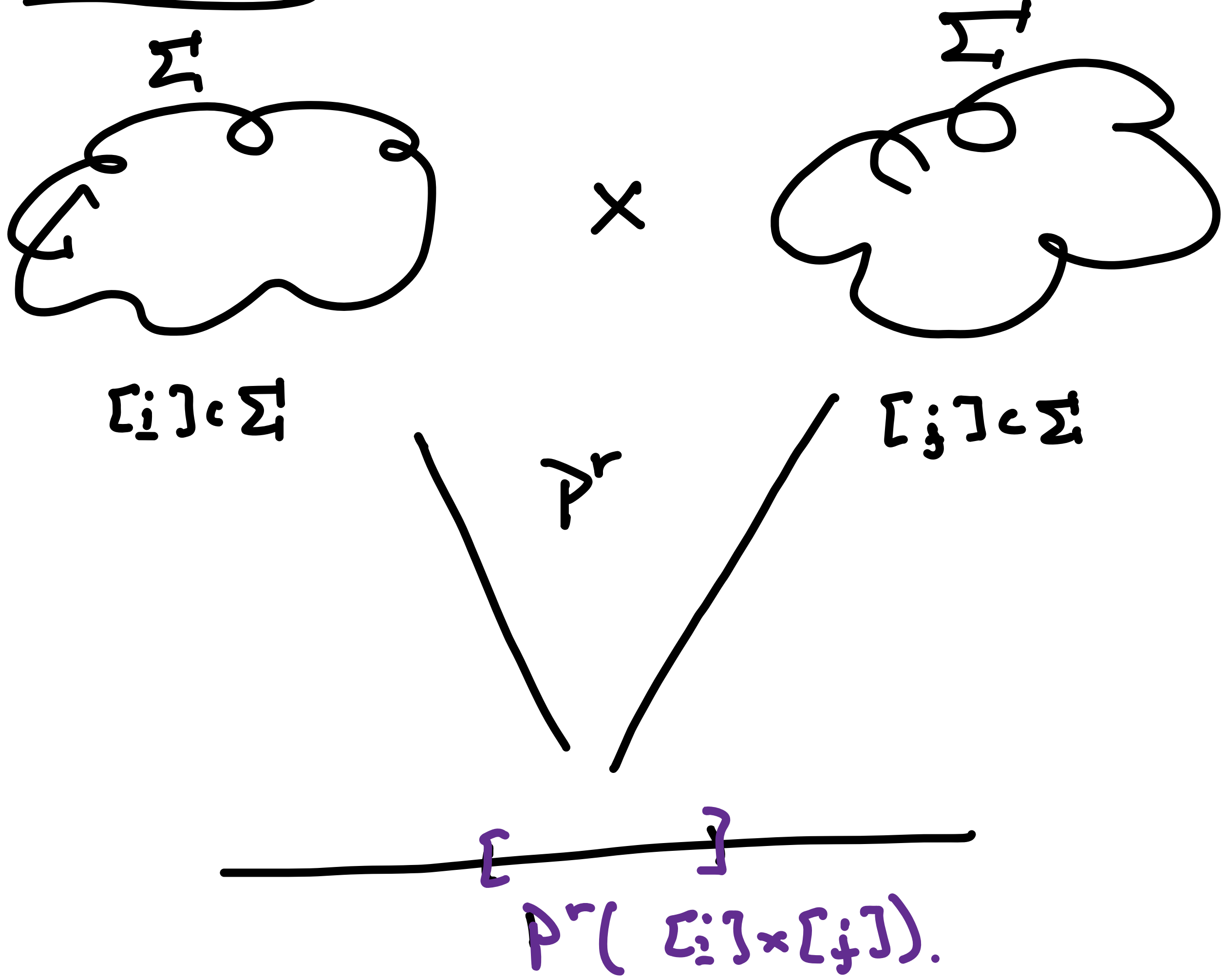
$$|P^r(\pi_n \underline{i}, \pi_n \underline{j}) - t| \lesssim \eta_n$$

For some given $(\eta_n)_{n=1}^{\infty}$ s.t. $\frac{1}{n}\log(\eta_n) \to -\infty$.

But for all $\underline{i}, \underline{j} \in \Sigma_1'$,

$$|P^r(\pi_n \underline{i}, \pi_n \underline{j}) - t| > 0.$$

Picture: Let $\underline{i}, \underline{j} \in \Sigma'_n$



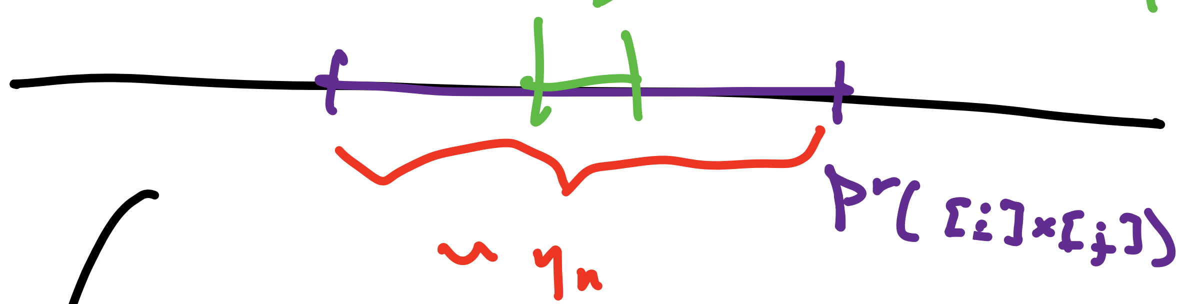
Idea: Jiggle r s.t.

$$\text{length}(P^r([\underline{i}] \times [\underline{j}])) \in [\tfrac{1}{2}\eta_n, 2\eta_n].$$

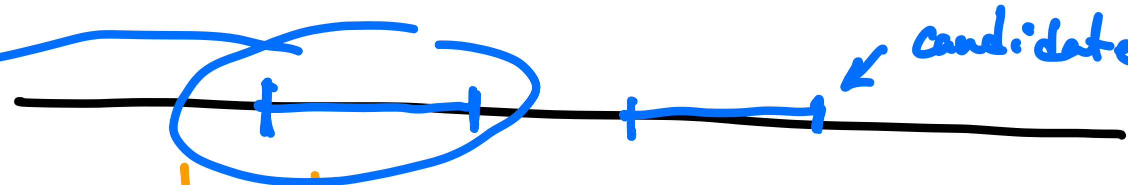
This is possible because $\{P^r\}_r$ is a transversal

Now we have

remove middle of size $> \frac{1}{100}\eta_n$.



candidate t's.



Now at the next step

$$P^r([i\ i_{n+1}] \times [j\ j_{n+1}]) \subset P^r([i] \times [j]).$$

again remove middle $\frac{1}{100}$th

Jiggle r again so that

$$P^r([i\ i_{n+1}] \times [j\ j_{n+1}]) \sim \eta_{n+1}$$

The amount of jiggles must converge.

We continue this process iteratively and get a fractal set of Hausdorff dimension zero.

Let's now prove the interesting proposition:

Prop: If $\Phi = (\phi_1, \phi_2, \ldots, \phi_m)$ is a homogeneous tuple of contractive similitudes acting on the real line such that r with $0 < |r| < \frac{1}{m}$ is the common contraction ratio of the maps, ϕ_i, $\Delta \subset \mathbb{R}$ is the associated self-similar set, and μ is the natural measure on X, then

$$\dim_{\mathcal{K}}(\mu) = \text{sim-dim}(\Phi) - \frac{X(\gamma)}{\log(|r|^{-1})}.$$

for all $0 < \gamma \leq |r|$.

Pf: As before, let

$$\nu^{(n)} := \sum_{\underline{i} \in \Sigma_n} p_{\underline{i}} \, \delta_{\phi_{\underline{i}}(c)}$$

since μ is the natural measure,

$$p_{\underline{i}} = \frac{1}{m^n}$$

$$\Rightarrow \quad \nu^{(n)} = \frac{1}{m^n} \sum_{\underline{i} \in \Sigma_n} \delta_{\phi_{\underline{i}}(c)}$$

Let $n' = n \dfrac{\log(\frac{1}{r})}{\log(2)}$

Thm: (Hochman)

$$\frac{1}{n'}\left(H(\nu^{(n)}, \mathcal{D}_{m_n}) - H(\nu^{(n)}, \mathcal{D}_{n'})\right) = 0$$

Therefore,

$$\dim_{\mathcal{H}}(\mu) = \lim_{n\to\infty} \frac{1}{n'} H(\mu, \mathcal{D}_{n'})$$

$$= \lim_{n\to\infty} \frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{n'})$$

$$= \lim_{n\to\infty} \frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{m_n})$$

$$= -\lim_{n\to\infty} \frac{1}{n'} \int \log_2\left(\nu^{(n)}(B(x, 2^{-m_n}))\right) d\nu^{(n)}$$

Therefore

$$\dim_{\mathcal{H}}(\mu) = -\lim_{n\to\infty} \frac{1}{n'} \sum_{\underline{i}\in I_n} \frac{1}{m^n} \int \log_2\left(\nu^{(n)}(B(x, 2^{-m_n})\right) \delta_{\phi_{\underline{i}}(0)}(dx)$$

$$= -\lim_{n\to\infty} \frac{1}{n'} \sum_{\underline{i}\in I_n} \frac{1}{m^n} \log_2\left(\nu^{(n)}\left(B(\phi_{\underline{i}}(0), 2^{-m_n})\right)\right)$$

$$= -\lim_{n\to\infty} \frac{1}{n'} \sum \frac{1}{m^n} \log_2\left(\frac{\#\{ j \in \Sigma_n \mid \phi_j(\omega) \in B(\phi_j(\omega), 2^{-M}) \}}{m^n} \right)$$

$$= \lim_{n\to\infty} \frac{1}{n'} \cdot \log_2(m^n) - X(2^{-M}).$$

$\square$.