Proof of Hochman's Thm

Let $\nu^{(n)} = \sum_{\underline{i} \in \Lambda_n} p_{\underline{i}} \cdot \delta_{\phi_{\underline{i}}(0)}$

and for A

$$\tau^{(n)}(A) = \mu(r^{-n}A)$$

Since $0 \in \Lambda$, $\mu = \nu^{(n)} * \tau^{(n)}$

Let $n' = n \frac{\log(1/r)}{\log(2)}$

Recall

$$\lim_{n\to\infty} \frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{n'}) = \lim_{n\to\infty} \frac{1}{n'} H(\mu, \mathcal{D}_{n'})$$

$$= \dim \mu = \alpha.$$

Let $M \in \mathbb{Z}_+$, $M >> 1$ be fixed. Then

$$\alpha \xleftarrow{n\to\infty} \frac{1}{Mn} H(\mu, \mathcal{D}_{Mn})$$

$$= \frac{n'}{Mn}\left(\frac{1}{n'} H(\mu, \mathcal{D}_{n'})\right) + \frac{Mn-n'}{Mn}\left(\frac{1}{Mn-n'} H(\mu, \mathcal{D}_{Mn} | \mathcal{D}_{n'})\right)$$

$$= \frac{\log(1/r)}{M\log(2)}\left(\frac{1}{n'}H(\mu, \mathcal{D}_{n'})\right) + \frac{M - \log(\frac{1}{r})}{M\log(2)}\left(\frac{1}{Mn-n'}H(\mu, \mathcal{D}_{Mn} | \mathcal{D}_{n'})\right)$$

α

Therefore,

$$\frac{1}{Mn-n'}H(\mu, \mathcal{D}_{Mn} | \mathcal{D}_{n'}) \xrightarrow{n\to\infty} \alpha$$

Note that $\nu^{(n)} = \mathbb{E}_{i=n'}(\nu^{(n)}_{y,i})$

$$\Rightarrow \mu = \nu^{(n)} * \tau^{(n)} = \mathbb{E}_{i=n'}[\nu^{(n)}_{y,i} * \tau^{(n)}]$$

Expectation with respect to $\nu^{(n)}$.

$\nu^{(n)}_{y,i} * \tau^{(n)}$ is supported on a set of size $O(2^{-n'})$ so

$$\left| H(\nu^{(n)}_{y,i} * \tau^{(n)}, \mathcal{D}_{Mn} | \mathcal{D}_{n'}) - H(\nu^{(n)}_{y,i} * \tau^{(n)} | \mathcal{D}_{Mn}) \right| = O(1).$$

Now by concavity of entropy

$$H(\mu, \mathcal{D}_{M_n} | \mathcal{D}_{n'}) = H(\nu^{(n)} * \tau^{(n)}, \mathcal{D}_{M_n} | \mathcal{D}_{n'})$$

$$\geq \mathbb{E}_{i=n'}\left[H(\nu^{(n)}_{y,i} * \tau^{(n)}, \mathcal{D}_{M_n} | \mathcal{D}_{n'})\right]$$

$$= \mathbb{E}_{i=n'}\left[H(\nu^{(n)}_{y,i} * \tau^{(n)}, \mathcal{D}_{M_n})\right] + O(1)$$

Thus,

$$\limsup_{n\to\infty} \frac{1}{M_n - n'} \mathbb{E}_{i=n'}\left[H(\nu^{(n)}_{y,i} * \tau^{(n)}, \mathcal{D}_{M_n})\right] \leq \alpha. \quad (*)$$

Note that

$$H(\tau^{(n)}, \mathcal{D}_{M_n}) = H(\mu, \mathcal{D}_{M_n - n'}) + O(1)$$

$\Rightarrow$

$$\lim_{n\to\infty} \frac{1}{M_n - n'} H(\tau^{(n)}, \mathcal{D}_{M_n}) = \lim_{n\to\infty} \frac{1}{M_n - n'} H(\mu, \mathcal{D}_{M_n - n'}) = \alpha.$$

$(**)$

Also,

$$\frac{1}{M_n - n'} H(\nu^{(n)}_{y,i} * \tau^{(n)}, \mathcal{D}_{M_n}) \geq \frac{1}{M_n - n'} H(\tau^{(n)}, \mathcal{D}_{M_n}) + O\left(\frac{1}{M_n - n'}\right)$$

(*) and (**) imply that $\forall \delta > 0$

$$\lim_{n\to\infty} \mathbb{P}_{i=n'}\left(\left| \frac{1}{M_{n-n'}} H(\nu_{y,i}^{(n)} * \tau^{(n)}, \mathcal{D}_{Mn}) - \alpha \right| < \delta \right) = 1$$

and thus

$$\lim_{n\to\infty} \mathbb{P}_{i=n'}\left(\left| \frac{1}{M_{n-n'}} H(\nu_{y,i}^{(n)} * \tau^{(n)}, \mathcal{D}_{Mn}) - \frac{1}{M_{n-n'}} H(\tau^{(n)}, \mathcal{D}_{Mn}) \right| < \delta \right) = 1.$$

For $\varepsilon > 0$ small enough m, n large enough

$$\mathbb{P}_{n' < i \le Mn}\left(H_m((\tau^{(n)})^{x,i}) < 1-\varepsilon \right)$$

$$\ge \mathbb{P}_{n' < i < Mn}\left(H_m((\tau^{(n)})^{x,i}) < \alpha + \varepsilon \right)$$

$$> 1 - \varepsilon.$$

Let y, i be such that

$$\left| \frac{1}{M_{n-n'}} H(\nu_{y,i}^{(n)} * \tau^{(n)}, \mathcal{D}_{Mn}) - \frac{1}{M_{n-n'}} H(\tau^{(n)}, \mathcal{D}_{Mn}) \right| < \delta$$

Then

$$\mathbb{P}_{n' \le i \le M_n}\left(H_m\left((\tau^{(n)})^{x,i} \right) < 1-\varepsilon \right)$$

$$> 1 - \varepsilon$$

and $\boxed{\text{Cor } A}$ implies

$$\frac{1}{M_n - n'} H\left(\nu^{(n)}_{y,i}, \mathcal{D}_{M_n}\right) < \varepsilon$$

Thus

$$\lim_{n \to \infty} \mathbb{P}_{i=n'}\left(\frac{1}{M_n - n'} H\left(\nu^{(n)}_{y,i}, \mathcal{D}_{M_n}\right) < \varepsilon \right) = 1$$

$\Rightarrow$

$$\lim_{n \to \infty} \frac{1}{M_n - n'} H\left(\nu^{(n)}, \mathcal{D}_{M_n} \mid \mathcal{D}_n\right)$$

$$= \lim_{n \to \infty} \frac{1}{M_n - n'} \mathbb{E}_{i=n'}\left[H\left(\nu^{(n)}_{y,i}, \mathcal{D}_{M_n}\right) \right]$$

$$= \lim_{n \to \infty} \mathbb{E}_{i=n'}\left[\frac{1}{M_n - n'} H\left(\nu^{(n)}_{y,i}, \mathcal{D}_{M_n}\right) \right]$$

$$< \varepsilon .$$

This concludes the proof $\square$.