Cor: Let $\sigma > 0$, $\delta > 0$, and let $m \geq 2$ be an integer. Then there exists an integer $p = p_0(\sigma, \delta, m)$ such that for all $k \geq k_0(\sigma, \delta, m)$, the following holds:

Let $\mu_1, \dots, \mu_k \in \mathcal{P}([0,1])$ and

$$\mu = \mu_1 * \mu_2 * \dots * \mu_k$$ and suppose

$\mathrm{Var}(\mu) \geq \sigma k$. Then

$$\mathbb{P}_{i = p - [\log(\sqrt{k})]}\left(\mu^{x,i} \text{ is } (\delta, m)\text{-uniform}\right) > 1 - \delta$$

Note that

$$\gamma_{mk, \sigma k} \approx \mathcal{L}^1\Big|_{[mk - \sqrt{k},\, mk + \sqrt{k}]}$$

$\Rightarrow$ Berry-Esseen implies

$$\mu \approx \mathcal{L}^1\Big|_{[mk - \sqrt{k},\, mk + \sqrt{k}]}.$$

$\Rightarrow$ $\mu^y \approx \mathcal{L}^1|_{[0,1]}$ $\quad y \in \mathcal{D}_n, \quad n \sim -\log(k^{1/2})$

Since the density function of

$\gamma_{mk,\sigma k}$ is continuous, for

p large enough, if $\xi' \in \mathcal{D}_{p+n}$

$$\left(\mu - \mathcal{L}^1\right)^{\xi'} \ll \left(\mu - \mathcal{L}^1\right)^{\xi} \lesssim_n K^{-1/2}.$$

Repeated Convolutions

Prop:

"If variance is large enough, then only k repeated convolutions are necessary so that $\underbrace{(\mu * \cdots * \mu)}_{k \text{ times}}\Big|_{[0,1]} \approx \mathcal{L}^1\Big|_{[0,1]}$"

Let $\sigma, \delta > 0$ and let $m \geq 2$, $m \in \mathbb{Z}_+$. Then $\exists p = p(\sigma, \delta, m)$ s.t. for sufficiently large $k = k(\sigma, \delta, m)$, the following holds:

Let $\mu \in \mathcal{P}([0,1])$ fix an integer

$i_0 \geq 0$ and write

$$\lambda = \mathbb{E}_{i=i_0}\left[\operatorname{Var}(\mu^{x,i}) \right].$$

If $\lambda > \sigma$, then for $j_0 = i_0 - \lfloor \log(\sqrt{k}) \rfloor + p$ and

$$\nu = \underbrace{\mu * \cdots * \mu}_{k \text{ times}} = \mu^{*k},$$

then

$$\mathbb{P}_{j=j_0}\left(\nu^{x,j} \text{ is } (\delta, m)\text{-uniform} \right) > 1 - \delta.$$

This is a statement that we will not prove. Part of the difficulty stems from the fact that

$$\nu^{x,j} = (\mu * \cdots * \mu) = \mu^{x_1,j} * \cdots * \mu^{x_k,j}$$

but Berry-Esseen applies in this case and we get uniformity. Beyond this application, the proof is straightforward.

Lemma: Fix $m \in \mathbb{N}$. If $\mathrm{Var}(\mu)$ is small enough then $H_m(\mu) \leq \frac{2}{m}$. If $H_m(\mu)$ is small enough, then $\mathrm{Var}(\mu) < 2^{-m}$

$\mathrm{Var}(\mu) \Leftrightarrow$ $\overset{\mu}{\text{concentrated}}$ $\Leftrightarrow H_m(\mu)$.

Corollary:

Let $m \in \mathbb{N}$ and $\varepsilon > 0$. For $N > N(m, \varepsilon)$ and $0 < \delta < \delta(m, \varepsilon, N)$ if $\mu \in \mathcal{P}([0,1])$ and $\mathrm{Var}(\mu) < \delta$, then

$$\mathbb{P}_{0 \leq i \leq n}\Big(\mathrm{Var}(\mu^{x,i}) < \varepsilon \text{ and } \mu^{x,i} \text{ is } (\varepsilon, m)\text{-atomic}\Big) > 1 - \varepsilon.$$

A straightforward application of local to global entropy if $\mathrm{Var}(\mu) <$ small

$\Rightarrow H_n(\mu) <$ small $\Rightarrow \mathbb{E}_{0 \leq i \leq n}\left[H_m(\mu^{x,i})\right] <$ small $+ O\left(\log\left(\frac{m}{n}\right)\right)$

$\Rightarrow \mathrm{Var}(\mu^{x,i}) <$ small most of the time for $m \ll n$.

Thm: Let $\delta > 0$ and $m \in \mathbb{Z}_+$, $m \geq 2$.
For $k \geq k_0(\delta, m)$ and for $n \geq n_0(\delta, m, k)$ the following holds:

For any $\mu \in \mathcal{P}([0,1])$, $\exists\, I, J \subset \{1, \dots, n\}$ disjoint s.t. $\#(I \cup J) > (1-\delta)n$ and if $\nu = \mu * \mu * \dots * \mu = \mu^{*k}$ then

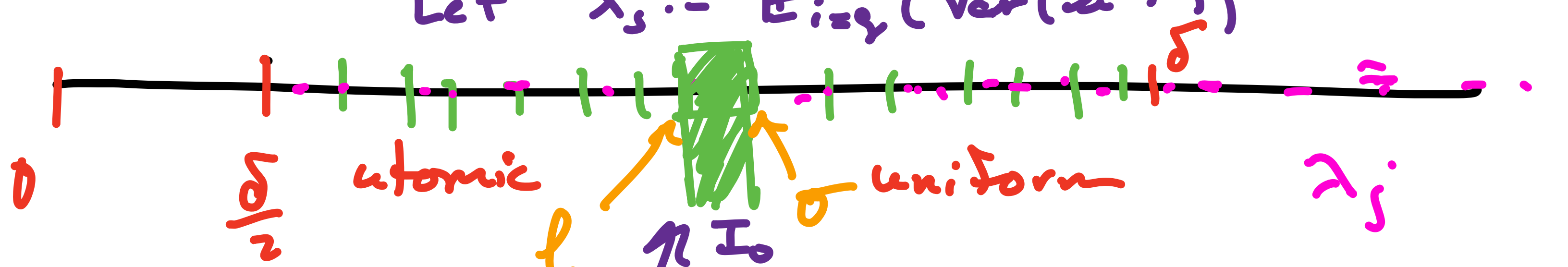
$$\mathbb{P}_{i=j}\left(\nu^{x,i} \text{ is } (\delta,m)\text{-uniform}\right) \geq 1-\delta \quad \text{for } j \in I$$

$$\mathbb{P}_{i=j}\left(\mu^{x,i} \text{ is } (\delta,m)\text{-atomic}\right) \geq 1-\delta \quad \text{for } j \in J$$

Proof idea

Split $[\frac{\delta}{2}, \delta]$ into $\sim 100\delta^{-1}$ pieces

Let $\lambda_j := \mathbb{E}_{i=j}(\mathrm{Var}(\mu^{x,i}))$



$$\mathbb{P}_{0 \le j \le n}(\lambda_j \in I_0) < \frac{\delta}{2}$$

Let $I' = \{ j \mid \lambda_j > \sigma \}$

$J' = \{ j \mid \lambda_j < \rho \}$

Clearly $\#(I' \cup J') \geq 1 - \frac{\delta}{2}$

For $k = k(\sigma)$ large enough
$= k(\frac{\delta}{2})$

and for ℓ large enough if $j \geq \ell$

and $j \in I'$,

$\mathbb{P}_{i=j}(v^{x,i}$ is (δ, m) uniform$) > 1 - \delta$.

For $j \in J'$,

$\mathbb{P}_{i=j}(\text{Var}(\mu^{x,i}) < \delta) > 1 - \delta$

For simplicity, let's say $\text{Var}(\mu^{x,i}) < \delta$

$\Rightarrow H_m(\mu^{x,i}) < \delta$ then $\mathbb{P}_{i=j}(\mu^{x,i}$ is (δ, m) atomic$) > 1 - \delta$

It suffices now to find n large enough so that $\#(J' \cup I' \cap [\ell, n])$

$$\geq (1-\delta)n$$

without breaking the conditions we've already established.
