IFS and Symbolic Space.

Let $\Phi = \{\phi_1, \dots, \phi_m\}$ be an IFS.

Define the symbolic space associated to Φ by

$$\Sigma := \{1, \dots, m\}^{\mathbb{N}}$$

For $\underline{i}, \underline{j} \in \Sigma$, define

$$\underline{i} \wedge \underline{j} = (i_1, i_2, \dots, i_n) = (j_1, j_2, \dots, j_n)$$

where $n = \sup\left\{ k \geq 1 \;\middle|\; \begin{matrix} i_m = j_m \text{ for all} \\ m \leq k \end{matrix} \right\}$.

Define the length of the common prefix, $\underline{i} \wedge \underline{j}$, by

$|\underline{i} \wedge \underline{j}| := n$ and define a metric on Σ by

$$\rho(\underline{i}, \underline{j}) := m^{-|\underline{i} \wedge \underline{j}|}$$

Let
$$\Sigma_1^* := \bigcup_{n=0}^{\infty} \Sigma_{1n}$$
with $\Sigma_{1n} := \{1, \dots, m\}^n$.

Symbolic Cylinders:
$$[i_1, \dots, i_n] := \{ \underline{j} \in \Sigma_1 \mid j_k = i_k \ \forall k \le n \}$$

Let $\mathcal{C} = \{\phi\} \cup \{ [i_1, \dots, i_n] \mid (i_1, \dots, i_n) \in \{1, \dots, m\}^n \}$

<u>Lemma:</u>
Let F be a function on Σ_1^* satisfying
$$F(\underline{i}) = \sum_{j=1}^{m} F(\underline{i}j) \quad \text{for all } \underline{i} \in \Sigma_1^*.$$
Then the set function, μ, on $\mathcal{C}$ defined by
$$\mu(\phi) = 0 \quad \text{and} \quad \mu([\underline{i}]) := F(\underline{i})$$
extends uniquely to a finite Borel measure on Σ_1

Example:

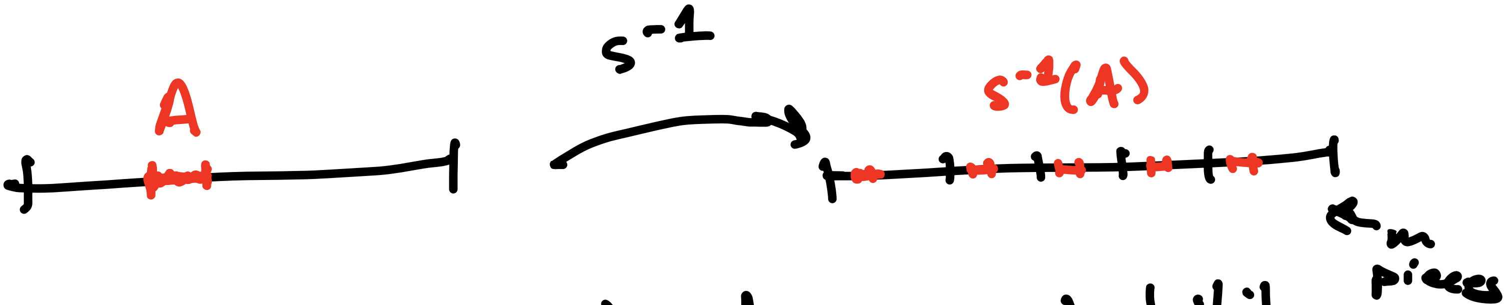
Shift invariant measure.

Let $s : \Sigma_1' \to \Sigma_1'$ be the left shift operator defined by

$$s(i_1, i_2, i_3, \ldots) := (i_2, i_3, \ldots).$$

A Borel measure, μ, is said to be <u>s-invariant</u> if

$$\mu(s^{-1}A) = \mu(A) \quad \text{for all Borel } A \subset \Sigma_1'$$



Let $(p_1, \ldots, p_m)$ be a probability vector. Then the function defined by

$$f : \Sigma_1'^* \to \Sigma_1'^*, \quad f(\emptyset) = 1.$$

$$f(\underline{i}j) = p_j f(\underline{i}) \quad \text{for all } \underline{i} \in \Sigma_1'^*$$

Then the extension of ϕ to Σ_1' is s-invariant.

Natural Projection

Given an IFS $\Phi = (\phi_1, \ldots, \phi_m)$ let $\Lambda = \Lambda_\Phi$ be the attractor of $\mathcal{F}$. The natural projection $\pi : \Sigma \to \Lambda$ is defined by

$$\pi(\underline{i}) := \lim_{n \to \infty} \phi_{i_1 i_2 i_3 \cdots i_n}(0)$$

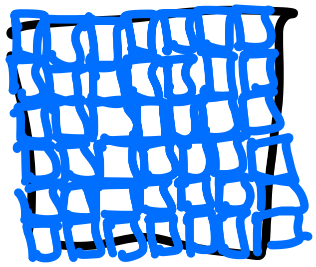
$$\Lambda = \bigcup_{\underline{i} \in \Sigma} \pi(\underline{i})$$

Define the level-n cylinders of the attractor Λ by

$$\Lambda_{i_1 \cdots i_n} := \phi_{i_1 \cdots i_n}(\Lambda).$$

Dimension

$\sim 2^k$ cubes of length 2^{-k} necessary to cover line



$\sim 2^{2k}$ cubes of length 2^{-k} necessary to cover cube.

This informs the following notion of dimension known as Minkowski or Box-Counting Dimension.

Dyadic cubes

For $k \in \mathbb{Z}$, define

$$\mathcal{D}_k := \left\{ \prod_{j=1}^{d} [2^{-k} m_j, 2^{-k}(m_j+1)) \;\middle|\; (m_1, \dots, m_d) \in \mathbb{Z}^d \right\}.$$

$\nwarrow$ Dyadic cubes at scale 2^{-k}.

Let $\mathcal{D} := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$.

<u>Definition</u> (Minkowski/Box Dimension)

Let $E \subset \mathbb{R}^d$. Then

$$\overline{\dim}_B(E) := \limsup_{k \to \infty} \frac{\log\left(\min\left\{\#\mathcal{E} \,\middle|\, E \subset \bigcup_{q \in \mathcal{E}} q,\ \mathcal{E} \subset \mathcal{D}_k\right\}\right)}{k \log 2},$$

$$\underline{\dim}_B(E) := \liminf_{k \to \infty} \frac{\log\left(\min\left\{\#\mathcal{E} \,\middle|\, E \subset \bigcup_{q \in \mathcal{E}} q,\ \mathcal{E} \subset \mathcal{D}_k\right\}\right)}{k \log 2}.$$

If $\overline{\dim}_B(E) = \underline{\dim}_B(E)$, then

$$\dim_B(E) := \overline{\dim}_B(E) = \underline{\dim}_B(E).$$

and we call $\dim_B(E)$ the Box counting or Minkowski dimension.

Alternative definition.

For $E \subset \mathbb{R}^d$, bounded and $\delta > 0$,

let $N(E,\delta) := \min\left\{ k \;\middle|\; E \subset \bigcup_{i=1}^{k} B(x_i,\delta) \text{ for some } \{x_j\} \subset \mathbb{R}^d \right\}$.

Then

$$\overline{\dim}_B(E) := \inf\left\{ s \;\middle|\; \limsup_{\varepsilon \to 0} N(E,\varepsilon)\varepsilon^s = 0 \right\}.$$

$$:= \inf\left\{ s \;\middle|\; \limsup_{\varepsilon \to 0} N(E,\varepsilon)\varepsilon^s < \infty \right\}.$$

$$:= \sup\left\{ s \;\middle|\; \limsup_{\varepsilon \to 0} N(E,\varepsilon)\varepsilon^s = \infty \right\}.$$

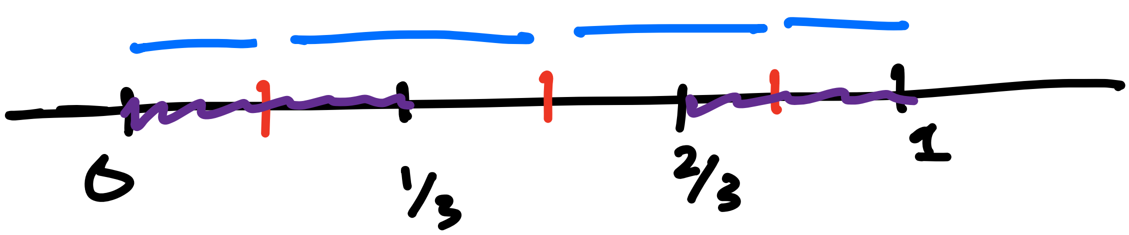
$$:= \sup\left\{ s \;\middle|\; \limsup_{\varepsilon \to 0} N(E,\varepsilon)\varepsilon^s > 0 \right\}.$$

and

$$\underline{\dim}_B(E) := \inf\left\{ s \;\middle|\; \liminf_{\varepsilon \to 0} N(E,\varepsilon)\varepsilon^s = 0 \right\}.$$

Example:

Consider the middle-third Cantor set, C.



At $k=2$, we need all dyadic intervals of the unit interval to cover the first stage of construction.

For $k >> 1$, we would need roughly $\left(\frac{2}{3}\right)^k \cdot 2^k$ intervals of length 2^{-k} to cover the k^{th} stage of construction.

Thus, the computation for dimension becomes

$$\frac{\log\left(\left(\frac{2}{3}\right)^k 2^k\right)}{k \log 2} = \frac{\log(4/3)}{\log(2)}$$

What if we used smaller intervals to avoid covering empty space.

For the k^{th} stage, we cover the construction intervals of length 2^{-Mk} for $M >> 1$.

Then we need roughly $\left(\frac{2}{3}\right)^k 2^{Mk}$ intervals to cover the kth stage of the construction.

Dimension computation:

$$\frac{\log\left(\left(2/3\right)^k 2^{Mk}\right)}{Mk \log 2} = \frac{(M+1)\log 2 - \log 3}{M \log 2}$$

$$\approx 1 \quad \text{for } M >> 1.$$

Instead, at the kth stage, find k' s.t. $2^{-k'}$ is approximately the size of a 3^{-k}-length interval.

$$\Rightarrow \quad k' \approx \frac{\log 3}{\log 2} k.$$

One needs $\sim 2^k$ intervals of this size to cover the kth stage.

Thus,

$$\frac{\log(2^k)}{\log(2^{k'})} = \frac{k \log 2}{\frac{\log 3}{\log 2} k \log 2} = \frac{\log 2}{\log 3}.$$

The alternative computation is easier

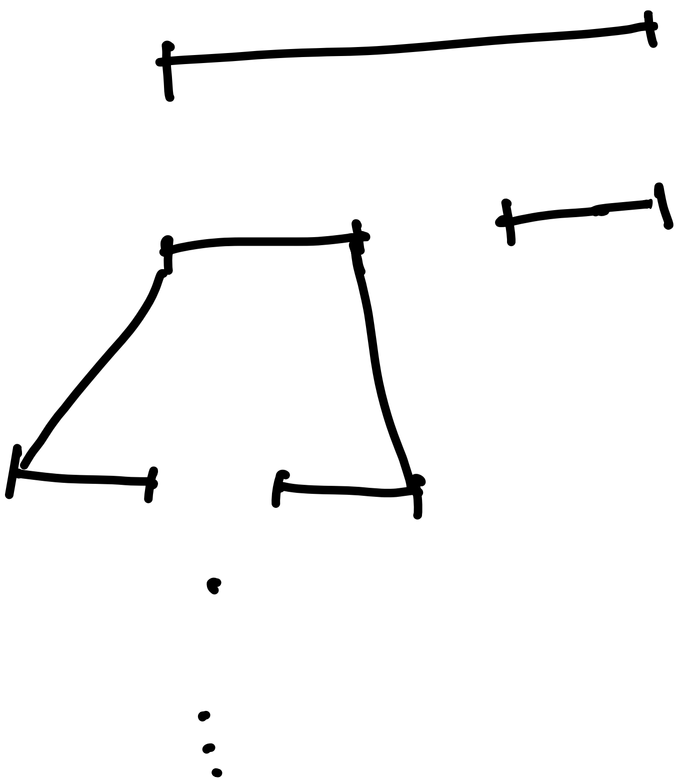
$$N(E, 3^{-k})(3^{-k})^s = 2^k (3^{-k})^s$$

$$\Rightarrow \quad \lim_{k \to \infty} N(E, 3^{-k})(3^{-k})^s \in (0, \infty)$$

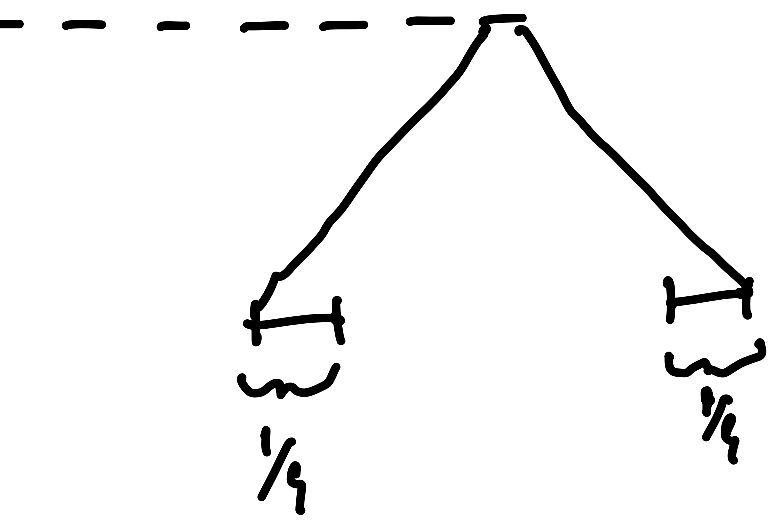
$$\Leftrightarrow \quad s = \frac{\log 2}{\log 3}.$$

Example:

For the first n_1 steps, construct a middle-third Cantor set



For the next n_2 steps construct middle-$7/9$ Cantor set



then the next n_3 steps, we switch back to middle-third Cantor set, then the next n_4 steps switch back to middle-7/9 Cantor set.

Then if

$$\frac{n_1 + \cdots + n_k}{n_{k+1}} \to 0$$

then
$$\underline{\dim}_B(\Lambda) = \frac{\log 2}{\log 9}$$

$$\overline{\dim}_B(\Lambda) = \frac{\log 2}{\log 3}.$$