Uniformity and Atomicity.

Let $\mu \in \mathcal{P}([0,1])$

Def: Let $\varepsilon > 0$, $m \in \mathbb{Z}_+$.

μ is (ε, m) atomic if

$$H_m(\mu) < \varepsilon$$

μ is (ε, m) uniform if

$$H_m(\mu) > 1 - \varepsilon$$

We are now prepared to state the inverse thm that is central to the proof of Hochman

<u>Thm</u> (Inverse Theorem for Entropy)

For every $\varepsilon_1, \varepsilon_2 > 0$ and integers $m_1, m_2 \geq 2$, there exists $\delta = \delta(\varepsilon_1, \varepsilon_2, m_1, m_2)$ such that for all $n > n(\varepsilon_1, \varepsilon_2, m_1, m_2, \delta)$ if $\mu, \nu \in \mathcal{P}([0,1])$, then either

- $$H_n(\mu * \nu) \geq H_n(\mu) + \delta$$

or

- $\exists$ disjoint subsets $I, J \subset \{0, \dots, n\}$ with $\#(I \cup J) \geq (1-\varepsilon) n$

  $\mathbb{P}_{i=k}(\mu^{x,i}$ is (ε_1, m_1)-uniform$) > 1-\varepsilon$ for $k \in I$

  $\mathbb{P}_{i=k}(\nu^{x,i}$ is (ε_2, m_2)-atomic$) > 1-\varepsilon$ for $k \in J$

Number Theory Analogy

First, let's return to measures

Suppose $\mu, \nu \in \mathcal{P}(\mathbb{T})$ where $\mathbb{R}/\mathbb{Z}$.

$$H_n(\mu * \nu) = H_n(\mu)$$

This holds when either

1.) $\nu = \delta_x$ for some $x \in \mathbb{T}$

2.) $\mu =$ Lebesgue measure.

<u>Q:</u> Is it true that

$$H_n(\mu * \nu) \approx H_n(\mu)$$

implies

either $\nu \approx \delta_x$

or $\mu \approx$ Lebesgue measure?

This question is analogous to the following number theoretic questions:

Let $A, B \subset \mathbb{Z}$

Let $A+B = \{a+b \mid a \in A,\ b \in B\}$

- What does $\#(A+B) \lesssim \#B$ imply about the structure of A and B?

- What does $\#(A+A) \lesssim \#A$ imply about the structure of A?

<u>Note</u>: For any $A \subset \mathbb{Z}$, $2(\#A) \leq \#(A+A) \leq (\#A)^2$

<u>Thm</u>: (Freiman (1973))

Let $A \subset \mathbb{Z}$, $K > 0$. If

$$\#(A+A) \leq K(\#A),$$

then A is contained in a proper arithmetic progression of "dimension" d_K and "size" $f_K \cdot \#A$.

<u>Def</u>: An arithmetic progression of dimension d and size L is a set of the form

$$P = \{ n_0 + l_1 n_1 + \cdots + l_d n_d \mid 0 \leq l_j \leq L_j \}$$

$n_j \in \mathbb{Z}$, $L = L_1 L_2 \cdots L_d$

P is <u>proper</u> if all the sums are distinct.

Essentially, controlling sumsets size controls the coset structure of a set

<u>Thm</u> (Green and Ruzsa (2006))

Generalized Freiman to all abelian groups.

<u>Thm</u> (Figalli + Jerison '16) (Generalized to measurable sets)

For $A \subset \mathbb{R}$, let $co(A)$ = convex hull of A.

$$\mathcal{L}^1(A+A) - 2\mathcal{L}^1(A) \geq \min\left(\mathcal{L}^1(co(A) \setminus A), \mathcal{L}^1(A)\right)$$

For $A \subset \mathbb{R}^d$,

$$\frac{\mathcal{L}^d(A+A) - 2\mathcal{L}^d(A)}{\mathcal{L}^d(A)} \geq_d \left(\frac{\mathcal{L}^d(co(A) \setminus A)}{\mathcal{L}^d(A)} \right)^{\alpha_d}.$$

.

Thm (Tao, '09)

For X, a random variable on an abelian group, G. Let X_1 and X_2 be independent copies of X.

and let

$$\sigma(X) = \exp\left(H(X_1 + X_2) - H(X) \right).$$

$\sigma(X) = 1$

iff

X is the uniform distribution on a coset of a finite subgroup of G.

(Tao also proves stability results as well)

Tools for the Inverse Thm

Let $\mu \in \mathcal{P}(\mathbb{R})$, let $\langle \mu \rangle = m(\mu)$ be the mean of μ

$$\langle \mu \rangle := \int x \, d\mu(x)$$

Define the variance by

$$\mathrm{Var}(\mu) = \int (x - \langle \mu \rangle)^2 \, d\mu(x).$$

If $\mu_1, \mu_2, \ldots, \mu_k \in \mathcal{P}(\mathbb{R})$

$$\langle \mu_1 * \mu_2 * \cdots * \mu_k \rangle = \sum_{i=1}^{k} \langle \mu_i \rangle$$

and

$$\mathrm{Var}(\mu_1 * \mu_2 * \cdots * \mu_k) = \sum_{i=1}^{k} \mathrm{Var}(\mu_i)$$

The Gaussian with mean m and variance σ^2 is given by

$$\gamma_{m,\sigma^2}(A) = \int_A \varphi\left(\frac{x-m}{\sigma}\right) dx,$$

normalized Gaussian to account for CLT.

where $\varphi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}|x|^2)$.

"Repeated convolutions converge to Gaussian"

Thm (Berry-Esseen?)

Let $\mu_1, \mu_2, \ldots, \mu_k$ be probability measures on $\mathbb{R}$ with finite third moments

$$\rho_i = \int |x|^3 \, d\mu_i(x).$$

Let $\mu = \mu_1 * \cdots * \mu_k$ and let

$$\gamma = \gamma_{\langle \mu \rangle, \mathrm{Var}(\mu)}.$$

Then for any interval, $I \subset \mathbb{R}$,

$$|\mu(I) - \gamma(I)| \lesssim \frac{\sum \rho_i}{\mathrm{Var}(\mu)^{3/2}}$$

In particular, if $\rho_i \leq C$ and $\sum_{i=1}^{k} \mathrm{Var}(\mu_i) \geq ck$ for constants $c, C > 0$, then

$$|\mu(I) - \gamma(I)| \lesssim O_{c,C}(k^{-1/2}).$$