We can now reduce Hochman's thm:

Hochman's Thm (Original Form)

Let Φ be a IFS on $\mathbb{R}$

For $n \in \mathbb{Z}_+$, we define

$$\Delta_n := \min\{ \operatorname{dist}(\Delta_{\underline{i}}, \Delta_{\underline{j}}) \mid \underline{i} \neq \underline{j},\ \underline{i}, \underline{j} \in \Sigma_n \}$$

If $\dim_\mu(\Delta) < \min(\text{sim-dim}(\Phi), 1)$

then $-\frac{1}{n} \log(\Delta_n) \xrightarrow{n \to \infty} \infty$

Hochman's Thm: (Second Form)

Let Φ be a IFS on $\mathbb{R}$,

with $p = (r_1^s, \dots, r_m^s)$.

$$\lim_{n \to \infty} \frac{H(\nu_p, \Delta_n)}{n \log(2)} < \min(\text{sim-dim}(\Phi), 1)$$

then

$$-\frac{1}{n} \log(\Delta_n) \xrightarrow{n \to \infty} \infty$$

Let's now focus on reducing the thm to an IFS with uniform contraction ratios.

The following is similar to what appears in Hochman's paper as well as a 2009 paper by Peres and Shmerkin

Prop: Let $\Phi = (\phi_i)_{i=1}^m$ be an IFS on $\mathbb{R}^d$ where $d=1$ or $d=2$ with attractor Λ. Then for all $\epsilon>0$, there exists an IFS $\Psi = (\psi_j)_{j=1}^k$ with attractor $\tilde{\Lambda}$ such that

- $\exists \rho \in (1,-1)\setminus\{0\}$ s.t. $\frac{d}{dx}\psi_j = \rho \quad \forall j$.
- $\text{sim-dim}(\Psi) > \text{sim-dim}(\Phi) - \epsilon$
- $\tilde{\Lambda} \subset \Lambda$
- For each $j \in \{1, \ldots, k\}$ there exist $\underline{i} \in \bigcup_{n=1}^{\infty} \{1, \ldots, m\}^n$ s.t. $\psi_j = \phi_{\underline{i}}$.

pf: We prove the $d=1$ case. In D-S '09, they use the commutativity of rotations for $d=2$.

In the $d=1$ case, we can characterize Φ in the following way:

For each $i \in \{1, \dots, n\}$ $\exists \ell_i \in (-1,1) \setminus \{0\}$, $x_i \in \mathbb{R}$ s.t.

$$\phi_i(x) = \ell_i x + x_i$$

Let $s = \text{sim-dim}(\Phi)$ and $r_i = |\ell_i|$

Then
$$\sum_{i=1}^{m} r_i^s = 1.$$

Let $\{e_1, \dots, e_m\} \subset \mathbb{R}^m$ be the standard basis of $\mathbb{R}^m$.

Consider the random walk which starts at 0 and moves from x to e_k with probability r_i^s

Let X_N = position of the random walk after N steps.

Then

$$E(X_N) = \sum_{i=1}^{m} N r_i^s e_i$$

Let

$$v = \sum_{i=1}^{m} \lceil N r_i^s \rceil e_i$$

and

$$\| v - E(X_N) \| \leq m^{1/2} .$$

Central limit thm now implies

$$\mathbb{P}\left(\left\{ \frac{X_N - v}{\sqrt{N}} \leq \frac{10\sqrt{m}}{\sqrt{N}} \right\} \right) \gtrsim \left(\int_0^{\sqrt{N}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-y^2}{2\sigma^2}} \, dy \right)^m$$

$$\gtrsim \left(\frac{\sqrt{N}}{\sqrt{m}} \right)^{-m}$$

for k large enough.

Let $v = (v_2, \ldots, v_m)$

To each $\underline{i} \in \{1, \ldots, m\}^N$

we associate a path in $\mathbb{Z}^d$,

$p_{\underline{i}} : \{0, 1, \ldots, N\} \rightarrow \mathbb{Z}^d$

where $p_{\underline{i}}(0) = 0$ and

if $i_j = l$ then $p_{\underline{i}}(j) - p_{\underline{i}}(j-1) = e_l$.

Since multiplication is commutative,

for each $\underline{i}$ s.t. $p_{\underline{i}}(N) = v$,

$$\phi_{\underline{i}}(x) = \rho x + z_{\underline{i}}$$

where
$$\rho = \prod_{j=1}^{m} \rho_j^{v_j}$$

and $z_{\underline{i}} \in \mathbb{R}$.

The probability that $p_{\underline{i}}(k) = v$
is equal to
$$\prod_{i=1}^{m} (r_i^s)^{v_i} \leq \prod_{i=1}^{m} r_i^{sNr_i^s}$$
Thus, if M_v is the number of paths s.t. $p_{\underline{i}}(N) = v$, then
$$M_v \geq c(\sqrt{N})^{-m} \left(\prod_{i=1}^{m} r_i^{sNr_i^s} \right)^{-1}.$$
for some constant c.

Let

$$\Psi = \{ \phi_{\underline{i}} \mid P_{\underline{i}}(N) = v \}$$

Then

if $\psi_j \in \Psi$, then $\exists z_j \in \mathbb{R}$ s.t.

$$\psi_j(x) = \rho x + z_j .$$

Since Ψ is uniformly contracting,

$$\text{sim-dim}(\Psi) = \frac{\log(M_v)}{\log(1/|\rho|)}$$

Since $|\rho| = \prod_{i=1}^{m} r_i^{v_i} \geq \prod_{i=1}^{m} r_i \prod r_i^{N r_i^s}$

$$\frac{\log(M_v)}{\log(1/|\rho|)} \geq \frac{\log(c) + (-m/2)\log(N) + N s \sum_{i=1}^{m} r_i^s \log(1/r_i)}{\sum \log(1/r_i) + N \sum r_i^s \log(1/r_i)}$$

$$\xrightarrow{N \to \infty} s.$$

$\square$.

We can now further reduce Hochman's thm:

Hochman's Thm: (Second Form)

Let Φ be a IFS on $\mathbb{R}$,
with $p = (r_1^s, \ldots, r_m^s)$.

$$\lim_{n\to\infty} \frac{H(\nu_p, \Delta_n)}{n \log(2)} < \min(\text{sim-dim}(\Phi), 1)$$

then

$$-\frac{1}{n} \log(\Delta_n) \xrightarrow{n\to\infty} \infty$$

Hochman's Thm (Third Form)

Let Φ be a uniformly contracting IFS on $\mathbb{R}$, with contraction $r \in (0,1)$.

If $$\lim_{n\to\infty} \frac{H(\nu_p, \Delta_n)}{n \log(2)} < \min\left(\frac{\log(m)}{\log(1/r)}, 1\right)$$

then $$\frac{-1}{n} \log(\Delta_n) \xrightarrow{n\to\infty} \infty.$$

Now we proceed with discretization:

For $n \in \mathbb{Z}_+$, $p = (p_1, \dots, p_m)$

define

$$\nu^{(n)} := \sum_{\underline{i} \in \Sigma^1_n} p_{\underline{i}} \cdot \delta_{\phi_{\underline{i}}(0)}$$

Then $\nu^{(n)} \longrightarrow \nu_p$ weakly and

if $n' = n \dfrac{\log(1/r)}{\log(2)}$

$$\lim_{n \to \infty} \frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{n'}) = \lim_{n \to \infty} \frac{1}{n} H(\nu_p, \mathcal{D}_n)$$

$$= \dim_{H}(\nu_p).$$

If, for each $\underline{i} \in \Sigma^1_n$, there is a

$q \in \mathcal{D}_{n'}$ s.t. $\phi_{\underline{i}}(0) \in q$

and $\phi_{\underline{j}}(0) \in q \Rightarrow \underline{j} = \underline{i}$ then

$$\frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{n'}) = \frac{1}{n' \log(2)} \cdot \sum_{\underline{i} \in \Sigma^1_n} p_{\underline{i}} \log(p_{\underline{i}})$$

$$= \frac{n \sum_{j=1}^{m} p_j \log(p_j)}{n \log(1/r)}$$

$$= \text{sim-dim}(\mu_p).$$

Thus, for any $m > n'$

if $\frac{1}{m} H(\nu^{(n)}, \mathcal{D}_m) < \text{sim-dim}(\mu_p)$

implies, there exists $q \in \mathcal{D}_m$

such that $\phi_{\underline{i}}(0), \phi_{\underline{j}}(0) \in q$ and $\underline{i} \neq \underline{j}$
$\underline{i}, \underline{j} \in \Sigma_n$.

$\Rightarrow |\phi_{\underline{i}}(0) - \phi_{\underline{j}}(0)| < 2^{-m}$.

We've already shown that

IF $\dim_{\mathcal{H}}(\nu_p) < \text{sim-dim}(\mu_p)$, then

$$\lim_{n \to \infty} \frac{1}{n'} H(\nu_p, \mathcal{D}_{n'}) = \lim_{n \to \infty} \frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{n'})$$

$$< \text{sim-dim}(\mu_p).$$

So it suffices to prove the following final reformulation of

Hochman's result:

Hochman's Thm (Final Form)

Let ν_p be a self-similar measure on $\mathbb{R}$ with uniform contraction ratios.

Let $\nu^{(n)} = \sum_{\underline{i} \in \Sigma_n} p_{\underline{i}} \, \delta_{\{\phi_{\underline{i}}(0)\}}$. If

$\dim_H(\nu_p) < 1$, then

$$\lim_{n \to \infty} \frac{1}{n'} H(\nu^{(n)}, \mathcal{D}_{qn'} \mid \mathcal{D}_{n'}) = 0$$

for all $q > 1$.

(Here $H(\nu^{(n)}, \mathcal{D}_{qn'} \mid \mathcal{D}_{n'}) = H(\nu^{(n)}, \mathcal{D}_{qn'}) - H(\nu^{(n)}, \mathcal{D}_{n'})$)

Now we approach a proof of this theorem