(Feng and Hu) Restated.

Thm: Let $p=(p_1,\dots,p_m)$, $\Phi=(\phi_1,\dots,\phi_m)$ be an IFS on $\mathbb{R}$. If μ_p on Σ is $\mu_p=\{p_1,\dots,p_m\}^{\mathbb{N}}$ and $\nu_p=\Pi_*\mu_p$. Then ν_p is exact dimensional and

$$\dim_H(\nu_p) = \frac{h_p - \left(-\int \log\left(\mu_{\underline{i}}^{Q}([i_1])\right) d\mu_p(\underline{i})\right)}{\chi_r^p}.$$

Pf:

Assume for simplicity that $\exists r$ s.t. $\phi_i(x)=rx+x_i$ for all $i\in\{1,\dots,m\}$.

Then $\chi_r^p = -\log(r)$.

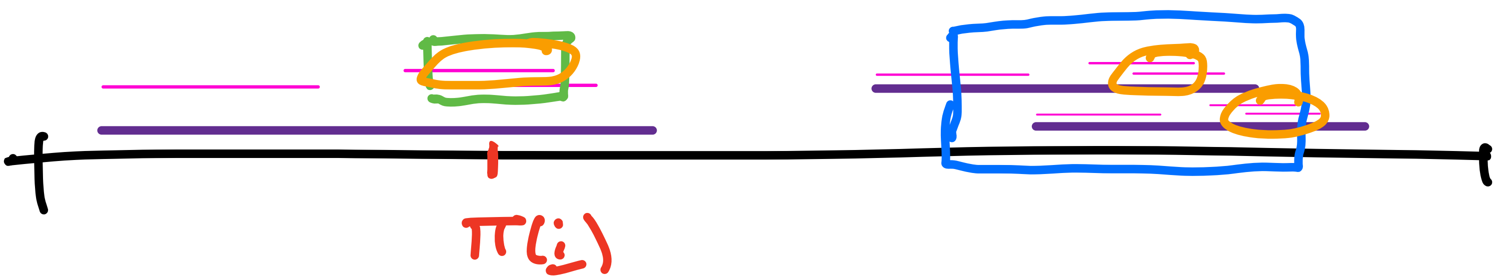
Also, assume $\Delta \subset B(0,\frac{1}{100})$.

For each $\underline{i} \in \Sigma_1$ and $n \in \mathbb{Z}_+$ we consider 3 subsets of Σ_1

$$\pi^{-1}(B(\pi(\underline{i}), r^n))$$

$$\pi^{-1}(B(\pi(\sigma \underline{i})), r^{n-1})$$

$$\sigma^{-1}\pi^{-1}(B(\pi(\sigma \underline{i})), r^{n-1})$$



<u>Note:</u>

$$\pi^{-1}(B(\pi(\sigma^l \underline{i}), r^{n-l})$$

$$= \pi^{-1}(B(\pi(\sigma^{l+1} \underline{i})), r^{n-l-1})$$

for $l \in \{0, \ldots, n-1\}$.

and since μ is shift invariant

$$\mu\left(\sigma^{-1}\pi^{-1}\left(B(\pi(\sigma\underline{i}), r^{n-1})\right)\right)$$
$$= \mu\left(\pi^{-1}\left(B(\pi(\sigma\underline{i}), r^{n-1})\right)\right)$$

Now we estimate the local dimension

$$\log(\nu(B(x,r^n))) = \log\left(\mu\left(\pi^{-1}(B(x,r^n))\right)\right)$$
$$= \sum_{k=0}^{n-1} \log\left(\mu\left(\pi^{-1}\left(B(\pi(\sigma^k\underline{i}), r^{n-k})\right)\right)\right) - \log\left(\mu\left(\pi^{-1}\left(B(\pi(\sigma^{k+1}\underline{i}), r^{n-k-1})\right)\right)\right)$$
$$\boxed{+\log\left(\mu\left(\pi^{-1}\left(B(\pi(\sigma^n\underline{i}), r^0)\right)\right)\right)} = 0$$

$$= \sum_{k=0}^{n-1} \log\left[\frac{\mu\left(\pi^{-1}\left(B(\pi(\sigma^k\underline{i}), r^{n-k})\right)\right)}{\mu\left(\pi^{-1}\left(B(\pi(\sigma^{k+1}\underline{i}), r^{n-k-1})\right)\right)}\right]$$

Each term in the sum gives some sense of the standard entropy and the amount of overlaps. In order to see this, we make the following observation:

Let $\underline{j} \in \Sigma_1^!$, $n \in \mathbb{Z}_+$.

Let $\mathcal{P}$ = partition of $\Sigma_1^!$
$= \{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$.

$\underline{i} \in \mathcal{P}_l$ if $i_1 = l$.

Let $\mathcal{P}(\underline{i}) = \{\underline{k} \in \Sigma_1^! \mid \underline{k} \in \mathcal{P}_{i_1}\}$.

Observe that

$$\sigma^{-1}\pi^{-1}(B(\pi(\sigma \underline{j}), r^{n-1})) \cap P(\underline{j})$$

$$= \sigma^{-1}\pi^{-1}(B(\pi(\sigma \underline{j}), r^{n-1})) \cap \pi^{-1}(B(\pi(\underline{j}), r^n))$$

$$= P(\underline{j}) \cap \pi^{-1}(B(\pi(\underline{j}), r^n))$$

Now

$$\log\left[\frac{\mu(\pi^{-1}(B(\pi(\underline{j}), r^n)))}{\mu(\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))}\right]$$

$$= \log\left[\frac{\mu(\pi^{-1}(B(\pi(\underline{j}), r^n)))}{\mu(\sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))}\right]$$

$$= \log\left(\frac{\mu\left(\mathcal{P}(\underline{j}) \cap \sigma^{-1}\pi^{-1}\left(B\left(\pi(\sigma \underline{j}), r^{n-1}\right)\right)\right)}{\mu\left(\sigma^{-1}\pi^{-1}\left(B\left(\pi(\sigma \underline{j}), r^{n-1}\right)\right)\right)} \right)$$

$$- \log\left(\frac{\mu\left(\mathcal{P}(\underline{j}) \cap \pi^{-1}\left(B\left(\pi(\underline{j}), r^{n}\right)\right)\right)}{\mu\left(\pi^{-1}\left(B\left(\pi(\underline{j}), r^{n}\right)\right)\right)} \right)$$

Now define

$$W_n(\underline{j}) := \log\left(\frac{\mu\left(\mathcal{P}(\underline{j}) \cap \sigma^{-1}\pi^{-1}\left(B\left(\pi(\sigma \underline{j}), r^{n-1}\right)\right)\right)}{\mu\left(\sigma^{-1}\pi^{-1}\left(B\left(\pi(\sigma \underline{j}), r^{n-1}\right)\right)\right)} \right)$$

$$G_n(\underline{j}) := \log\left(\frac{\mu\left(\mathcal{P}(\underline{j}) \cap \pi^{-1}\left(B\left(\pi(\underline{j}), r^{n}\right)\right)\right)}{\mu\left(\pi^{-1}\left(B\left(\pi(\underline{j}), r^{n}\right)\right)\right)} \right)$$

Now

$$\frac{1}{n} \log\left(\nu\left(B\left(\pi(\underline{i}), r^n\right)\right)\right)$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} W_{n-k}(\sigma^k \underline{i}) - \frac{1}{n} \sum_{k=0}^{n-1} G_{n-k}(\sigma^k \underline{i})$$

We need the following proposition and some intuition from our example to finish the proof:

Proposition (Special Case of Maker's Ergodic Theorem)

Let m be an ergodic σ-invariant measure and let $(f_k)_{k\in\mathbb{Z}_+} \subset L^1(\Sigma, m)$ be a sequence which converges both in L^1 and pointwise a.e. to $f \in L^1(\Sigma, m)$ then for a.e. $\underline{j} \in \Sigma$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} f_{n-l}(\sigma^l \underline{j}) = \int f(\underline{i})\, dm(\underline{i})$$

We won't prove this proposition

Assuming this proposition it suffices to show that

$$\lim_{n\to\infty} W_n(\underline{i}) = \sum_{l=1}^{m} \chi_{[l]}(\underline{i}) \log(m([l]))$$

and

$$\lim_{n\to\infty} G_n(\underline{i}) = \log(\mu^{Q}_{\underline{i}}([i_1]))$$

For W_n, recall that

$$W_n(j) := \log\left(\frac{\mu(A(j) \cap \sigma^{-1}\pi^{-1}(B(\pi(\sigma j), r^{n-1})))}{\mu(\sigma^{-1}\pi^{-1}(B(\pi(\sigma j), r^{n-1})))} \right)$$

Note that

$A(j) = [l]$ for some $l \in \{1, \dots, m\}$

and for any $E \in \Sigma_1$,

$$\mu([l] \cap \sigma^{-1}(E)) = \mu([l])\,\mu(\sigma^{-1}(E)).$$

Thus,

$$W_n(j) = \log\left(\frac{\mu(A(j))\,\mu(\sigma^{-1}\pi^{-1}(B(\pi\sigma j, r^{n-1})))}{\mu(\sigma^{-1}\pi^{-1}(B(\pi\sigma j, r^{n-1})))} \right)$$

$$= \log(\mu(A(j)))$$

$$= \sum_{l=1}^{m} \chi_{[l]}(j) \log(\mu([l]))$$

For G_n

$$\frac{\mu\left(P(\underline{j}) \cap \pi^{-1}\left(B(\pi_{\underline{j}}, r^n)\right)\right)}{\mu\left(\pi^{-1}\left(B(\pi_{\underline{j}}, r^n)\right)\right)}$$

$$= \frac{\int_{\pi^{-1}(B(\pi_{\underline{j}}, r^n))} \mu_{\underline{i}}^{Q}(P(\underline{j}))\, d\mu(\underline{i})}{\mu\left(\pi^{-1}\left(B(\pi_{\underline{j}}, r^n)\right)\right)}$$

$$\xrightarrow[r \to 0]{\text{Radon-Nikodym}} \mu_{\underline{j}}^{Q}(P(\underline{j})) \quad \text{for a.e. } \underline{j}.$$

$\square$

Another Notion of measure dimension.

Let μ be a prob. measure on a metric space, X, the <u>r-scale entropy</u> is defined as

$$H_r(\mu) = -\int \log(\mu(B(x,r)))\ \mu(dx)$$

The entropy dimension of μ is

$$\dim_e(\mu) = \lim_{r\to 0} \frac{H_r(\mu)}{-\log r}.$$

if the limit exists

<u>Prop:</u> (Fan, Lau, and Rao)

$$\operatorname*{ess\,inf}_{\mu} \liminf_{r\to 0} \frac{\log(\mu(B(y,r)))}{-\log(r)}$$

$$\leq \liminf_{r\to 0} \frac{H_r(\mu)}{-\log(r)}$$

$$\leq \limsup_{r\to 0} \frac{H_r(\mu)}{-\log(r)} \leq \operatorname*{ess\,sup}_{r\to 0} \frac{\log(\mu(B(y,r)))}{-\log(r)}.$$

Obvious

(Peres - Solomyak)

Prop: If the limit exists

$$\dim_e(\mu) = \lim_{n\to\infty} \frac{H(\mu, \mathcal{D}_n)}{n \log(2)}$$

Straightforward.

Recall that $H(\mu, \mathcal{D}_n) = -\sum_{q \in \mathcal{D}_n} \mu(q) \log(\mu(q))$

where $\mathcal{D}_n = \{\text{dyadic cubes}\}$

Corollary to Props (Young '82)

If μ is compactly supported and exact dimensional, then

$$\dim_e(\mu) = \lim_{n\to\infty} \frac{H(\mu, \mathcal{D}_n)}{n \log 2} = \dim_H(\mu)$$

(Note: Young proved this statement ≈ 20 years before the propositions were proven)

(*) Corollary to Young '82 and Feng-Hu '09.

Let $p = (p_1, \dots, p_m)$ be a probability vector and let $\Phi = (\phi_i)_{i=1}^m$ be an IFS on $\mathbb{R}$. If $\mu_p = \{p_1, \dots, p_m\}^{\mathbb{N}}$ and $\nu_p = \pi_* \mu_p$ then

$$\dim_H(\nu_p) = \lim_{n \to \infty} \frac{H(\nu_p, D_n)}{n \log(2)}.$$

We can now reduce Hochman's thm:

Hochman's Thm (Original Form)

Let Φ be a IFS on $\mathbb{R}$

For $n \in \mathbb{Z}_+$, we define

$$\Delta_n := \min\{ \mathrm{dist}(\Lambda_{\underline{i}}, \Lambda_{\underline{j}}) \mid \underline{i} \neq \underline{j}, \underline{i}, \underline{j} \in \Sigma_n' \}$$

If $\dim_H(\Lambda) < \min(\text{sim-dim}(\Phi), 1)$

then $-\frac{1}{n}\log(\Delta_n) \xrightarrow{n\to\infty} \infty$

Hochman's Thm: (Second Form)

Let Φ be a IFS on $\mathbb{R}$,

with $p = (r_1^s, \dots, r_m^s)$.

$$\lim_{n\to\infty} \frac{H(\nu_p, \Delta_n)}{n\log(2)} < \min(\text{sim-dim}(\Phi), 1)$$

then

$$-\frac{1}{n}\log(\Delta_n) \xrightarrow{n\to\infty} \infty$$