(Feng and Hu) Restated.

Thm: Let $p = (p_1, \ldots, p_m)$, $\Phi = (\phi_1, \ldots, \phi_m)$ be an IFS on $\mathbb{R}$. If μ_p on Σ is $\mu_p = \{p_1, \ldots, p_m\}^{\mathbb{N}}$ and $\nu_p = \Pi_* \mu_p$. Then ν_p is exact dimensional and

$$\dim_H(\nu_p) = \frac{h_p - \left(-\int \log\left(\mu_{\underline{i}}^{Q}([i_1])\right) d\mu_p(\underline{i})\right)}{\chi_r^p}.$$

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pf:

Assume for simplicity that $\exists r$ s.t.

$\phi_i(x) = rx + x_i$ for all $i \in \{1, \ldots, m\}$.

Then $\chi_r^p = -\log(r)$.

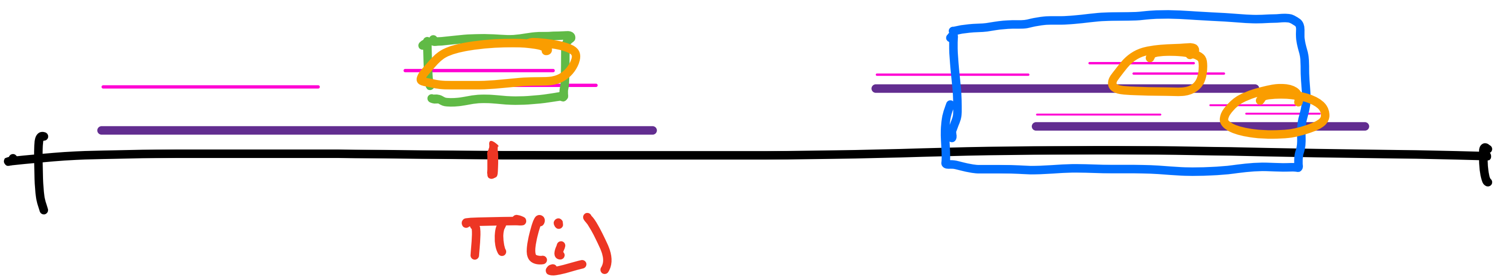
Also, assume $\Delta \subset B(0, \frac{1}{100})$.

For each $\underline{i} \in \Sigma_1$ and $n \in \mathbb{Z}_+$ we consider 3 subsets of Σ_1

$$\pi^{-1}(B(\pi(\underline{i}), r^n))$$

$$\pi^{-1}(B(\pi(\sigma \underline{i})), r^{n-1})$$

$$\sigma^{-1}\pi^{-1}(B(\pi(\sigma \underline{i})), r^{n-1})$$



<u>Note:</u>

$$\pi^{-1}(B(\pi(\sigma^{l}\underline{i}), r^{n-l})$$

$$= \pi^{-1}(B(\pi(\sigma^{l+1}\underline{i})), r^{n-l-1})$$

for $l \in \{0, \ldots, n-1\}$.

and since μ is shift invariant

$$\mu\left(\sigma^{-1}\pi^{-1}\left(B\left(\pi(\sigma\underline{i}),r^{n-1}\right)\right)\right)$$

$$=\mu\left(\pi^{-1}\left(B\left(\pi(\sigma\underline{i}),r^{n-1}\right)\right)\right)$$

Now we estimate the local dimension

$$\log\left(\nu(B(x,r^n))\right)=\log\left(\mu\left(\pi^{-1}\left(B(x,r^n)\right)\right)\right)$$

$$=\sum_{k=0}^{n-1}\log\left(\mu\left(\pi^{-1}\left(B\left(\pi(\sigma^k\underline{i}),r^{n-k}\right)\right)\right)\right)$$

$$-\log\left(\mu\left(\pi^{-1}\left(B\left(\pi(\sigma^{k+1}\underline{i}),r^{n-k-1}\right)\right)\right)\right)$$

$$\underbrace{+\log\left(\mu\left(\pi^{-1}\left(B\left(\pi(\sigma^n\underline{i}),r^0\right)\right)\right)\right)}_{=0}$$

$$=\sum_{k=0}^{n-1}\log\left[\frac{\mu\left(\pi^{-1}\left(B\left(\pi(\sigma^k\underline{i}),r^{n-k}\right)\right)\right)}{\mu\left(\pi^{-1}\left(B\left(\pi(\sigma^{k+1}\underline{i}),r^{n-k-1}\right)\right)\right)}\right]$$

Each term in the sum gives some sense of the standard entropy and the amount of overlaps. In order to see this, we make the following observation:

Let $\underline{j} \in \Sigma_1^!$, $n \in \mathbb{Z}_+$.

Let $\mathcal{P}$ = partition of $\Sigma_1^!$
$= \{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$.
$\underline{i} \in \mathcal{P}_l$ if $i_1 = l$.

Let $\mathcal{P}(\underline{i}) = \{\underline{k} \in \Sigma_1^! \mid \underline{k} \in \mathcal{P}_{i_1}\}$.

Observe that

$$\sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})) \cap P(\underline{j})$$

$$= \sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})) \cap \pi^{-1}(B(\pi(\underline{j}), r^n))$$

$$= P(\underline{j}) \cap \pi^{-1}(B(\pi(\underline{j}), r^n))$$

Now

$$\log\left[\frac{\mu(\pi^{-1}(B(\pi(\underline{j}), r^n)))}{\mu(\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))}\right]$$

$$= \log\left[\frac{\mu(\pi^{-1}(B(\pi(\underline{j}), r^n)))}{\mu(\sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))}\right]$$

$$= \log\left(\frac{\mu\left(P(\underline{j}) \cap \sigma^{-1}\pi^{-1}(B(\pi(\sigma \underline{j}), r^{n-1}))\right)}{\mu(\sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))} \right)$$

$$- \log\left(\frac{\mu\left(P(\underline{j}) \cap \pi^{-1}(B(\pi(\underline{j}), r^{n}))\right)}{\mu(\pi^{-1}(B(\pi(\underline{j}), r^{n})))} \right)$$

Now define

$$W_n(\underline{j}) := \log\left(\frac{\mu\left(P(\underline{j}) \cap \sigma^{-1}\pi^{-1}(B(\pi(\sigma \underline{j}), r^{n-1}))\right)}{\mu(\sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))} \right)$$

$$\underline{G_n(\underline{j}) := \log\left(\frac{\mu\left(P(\underline{j}) \cap \pi^{-1}(B(\pi(\underline{j}), r^{n}))\right)}{\mu(\pi^{-1}(B(\pi(\underline{j}), r^{n})))} \right)}$$

Now

$$\frac{1}{n}\log(\nu(B(\pi(\underline{i}), r^n)))$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} W_{n-k}(\sigma^k \underline{i}) - \frac{1}{n}\sum_{k=0}^{n-1} G_{n-k}(\sigma^k \underline{i})$$

We need the following proposition and some intuition from our example to finish the proof:

Proposition (Special Case of Maker's Ergodic Theorem)

Let m be an ergodic σ-invariant measure and let $(f_k)_{k \in \mathbb{Z}_+} \subset L^1(\Sigma, m)$ be a sequence which converges both in L^1 and pointwise a.e. to $f \in L^1(\Sigma, m)$ then for a.e. $\underline{j} \in \Sigma$:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=0}^{n-1} f_{n-l}(\sigma^l \underline{j}) = \int f(\underline{i}) \, dm(\underline{i})$$

We won't prove this proposition

Assuming this proposition it suffices to show that

$$\lim_{n \to \infty} W_n(\underline{i}) = \sum_{l=1}^{m} x_{[l]}(\underline{i}) \log(m([l]))$$

and

$$\lim_{n\to\infty} G_n(\underline{i}) = \log(\mu^Q_{\underline{i}}([i_1]))$$

For W_n, recall that

$$W_n(\underline{j}) := \log\left(\frac{\mu(P(\underline{j}) \cap \sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))}{\mu(\sigma^{-1}\pi^{-1}(B(\pi(\sigma\underline{j}), r^{n-1})))}\right)$$

Note that

$$P(\underline{j}) = [l] \quad \text{for some} \quad l \in \{1, \dots, m\}$$

and for any $E \in \Sigma_1$,

$$\mu([l] \cap \sigma^{-1}(E)) = \mu([l])\,\mu(\sigma^{-1}(E)).$$

Thus,

$$W_n(\underline{j}) = \log\left(\frac{\mu(P(\underline{j}))\,\mu(\sigma^{-1}\pi^{-1}(B(\pi\sigma\underline{j}, r^{n-1})))}{\mu(\sigma^{-1}\pi^{-1}(B(\pi\sigma\underline{j}, r^{n-1})))}\right)$$

$$= \log(\mu(P(\underline{j})))$$

$$= \sum_{l=1}^{m} \chi_{[l]}(\underline{j}) \log(\mu([l]))$$

For G_n

$$\frac{\mu\left(P(j) \cap \pi^{-1}(B(\pi_j, r^n))\right)}{\mu\left(\pi^{-1}(B(\pi_j, r^n))\right)}$$

$$= \frac{\displaystyle\int_{\pi^{-1}(B(\pi_j, r^n))} \mu_{\underline{i}}^{Q}(P(j))\, d\mu(\underline{i})}{\mu\left(\pi^{-1}(B(\pi_j, r^n))\right)}$$

$$\xrightarrow[r \to 0]{\text{Radon-Nikodym}} \mu_j^{Q}(P(j)) \quad \text{for a.e. } j.$$

$\square$